Gödelian Explorations

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Preface

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1 Wittgenstein's *Tractatus* in Brief: *Remarks on the Scaffolding*

> ”Mr. Wittgenstein's *Tractatus Logico-Philosophicus,* whether or not it proves to give the ultimate truth on the matters with which it deals, certainly deserves, by its breadth and scope and profundity, to be considered an important event in the philosophical world.”
>
> —Bertrand Russell, Introduction to the *Tractatus*

> ”The book deals with the problems of philosophy, and shows, I believe that the reason why these problems are posed is that the logic of our language is misunderstood. The whole sense of the book might be summed up in the following words: *what can be said at all can be said clearly, and what we cannot talk about we must pass over in silence.*”
>
> —Ludwig Wittgenstein, Preface to the *Tractatus*

Wittgenstein's philosophy of logic in the *Tractatus* was the result of his study *of* Gottlob Frege's writings and study *with* Bertrand Russell on the philosophy of mathematics. Frege's and Russell's work arose from their *logicism*—an attempt to construct and secure the foundations of mathematics, especially arithmetic, on the basis of logic and set theory. The thesis of *logicism* can be set forth as follows:

$$\text{Mathematics} = \text{Logic} + \text{Set Theory}.$$

Wittgenstein's *Tractatus* is inspired, not so much by logical or mathematical rigor, but rather by a sense of the austere and aesthetic beauty of formal logic. Wittgenstein's goal was to construct a philosophy of logic that could be reconciled with mathematics, science, probability, morality, the self, and his belief in a mystical worldview.

Wittgenstein's Kantian aim in the *Tractatus* is to understand the structure and limits of thought by examining the structure and limits of language. The limits of language are determined by its internal logical structure. Using his theory of logic, Wittgenstein places the propositions of science as within the limits of sense but places religion and morality beyond those limits. The logical positivists use this demarcation line to abandon the claims of religion, morality, and aesthetics as cognitively meaningless, but for Wittgenstein, this demarcation line protected the realms of religion, morality and aesthetics from intrusion of scientific verificationism. According to Wittgenstein, philosophical troubles arise from the desire to transcend the world of human thought and experience and trying to adopt an Archimedean standpoint. The over-arching architectonic or formal structure of the *Tractatus* can be seen from its whole numbered propositions:

1	The world is all that is the case.
2	What is the case — a fact — is the existence of states of affairs.
3	A logical picture of facts is a thought.
4	A thought is a proposition with a sense.
5	A proposition is a truth-function of elementary propositions.
6	The general form of a proposition is $[\bar{p}, \bar{\xi}, N(\bar{\xi})]$. This is the general form of a proposition.

	6.54	My propositions serve as elucidations in the following way: anyone who understands me eventually recognizes them as nonsensical, when he has used them—as steps—to climb up beyond them. (He must, so to speak, throw away the ladder after he has climbed up it.)
7		What we cannot speak about we must pass over in silence.

The genesis of the *Tractatus* is the grand statement that the world is all that is the case. And then it states that what is the case, a fact, is characterized as the existence of a state of affairs. A proposition asserts the existence of a state of affairs. Which is a logical picture of facts or a thought. The flow of the *Tractatus* continues in this way by starkly elegant propositions connecting one term to the next until one reaches the famous claim that the propositions of the *Tractatus* themselves are only the rungs of a ladder that must be kicked away. Wittgenstein's dichotomy that logical relations can only be "shown" not "said" casts doubt on the meaningfulness of the previous propositions, which, after all, attempt to say a great deal about logic. The *Tractatus* ends with the mystical admonition that "*of that we cannot speak, we must consign to silence.*"

Rudolf Carnap (1891 - 1970), who was described by the American logician W. V. O. Quine as the "embodiment of logical positivism, logical empiricism, the Vienna Circle,"[1] gives a vivid portrait of Wittgenstein's manner of philosophizing:

> His point of view and his attitude towards people and problems, even theoretical problems, were much more similar to those of a creative artist than to those of a scientist; one might almost say, similar to those of a religious prophet or a seer. When he started to formulate his view on some specific philosophical problem, we often felt the internal struggle that occurred to him at that very moment, a struggle by which he tried to penetrate from darkness to light under an intense and painful strain, which was even visible on his most expressive face. When finally, sometimes after a prolonged arduous effort, his answer came forth, his statement stood before us like a newly created piece of art or a divine revelation. Not that he asserted his views dogmatically… But the impression he made on us was as if insight came to him as through a divine inspiration, so that we could not help feeling that any sober rational comment or analysis of it would be profanation.

[1] Quine's 1970 Tribute to Carnap.

The goal of this short tour of the *Tractatus* is to give you a sense of some of its main doctrines and to point out its clashing conception of logic found in the works of Gödel and Turing. Although Gödel and Wittgenstein never met, it is clear that they were very aware of the reputation of each other work. Wittgenstein writes in his *Remarks on the Foundations of Mathematics*:

> Let us suppose I prove the unprovability (in Russell's system) of *P*; then by this proof I have proved *P*. Now if this proof were one in Russell's system — I should in this case have proved at once that it belonged and did not belong to Russell's system. — That is what comes of making up such sentences. But there is a contradiction here! – Well, then there is a contradiction here. Does it do any harm here? p. 51e

In a letter to his friend Karl Menger, who had organized a mathematics colloquium which branches off from the Vienna Circle and which was less interested in dismissing religious claims with more of an emphasis on mathematics, Gödel wrote:

"I also read parts of it [Wittgenstein's *Remarks on the Foundations of Mathematics*]. It seemed to me at the time that the benefit created by it may be mainly that it shows the falsity of the assertions set forth in it" and in a footnote added: "and in the *Tractatus* (the book itself contains very few assertions)." [2]

[2] "Gödel's remarks about Wittgenstein are cited by Solomon Feferman, the editor-in-chief of the monumental Gödel's Collected Works, as a"gem" in his reflections in "The Gödel Editorial Project: A Synopsis", reprinted in Kurt Gödel: Essays for His Centennial, edited by Feferman, Parsons, and Simpson (Cambridge University Press), p. 11.

There is a growing literature on Wittgenstein's "notorious passage" about Gödel Incompleteness addressing the question whether Wittgenstein understood Gödel's theorems and their significance.

References

Juliet Floyd and Hilary Putnam, "A Note on Wittgenstein's 'Notorious Paragraph' about the Gödel theorem," *Journal of Philosophy*, vol. 97 (2000): 624–632. Timothy Bays, "On Floyd and Putnam

Logical Atomism: Facts (1 – 1.21), States of Affairs (2 – 2.0141); Objects (2.02 – 2.063)

1	The world is all that is the case.

Note

Suppose that P and Q are atomic facts. Then one logical fact composed from these logical atoms could be expressed by something like this: P if and only if Q is *equivalent* to saying *not both P and Q only if neither P nor Q*. Here the notion of equivalence is *tautological*. Given the truth of the biconditional $P \iff Q$ is would be incorrect to say P is *equivalent* to Q. The *biconditional* $P \iff Q$ is true if either both P and Q are true or both P and Q are false. $(P \wedge Q)$ is *tautologically equivalent* to $(\neg P \rightarrow Q)$ because $[(P \wedge Q) \iff (\neg P \rightarrow Q)]$ is a tautology. Similarly, given the truth of the conditional $P \iff Q$ it would be incorrect to say P *implies* Q. The *conditional* $(PtoQ)$ is true whenever P happens to be true or Q happens to be false. $\neg P$ *tautologically implies* $(P \rightarrow Q)$ because the conditional $\neg P \rightarrow (P \rightarrow Q)$ is a tautology.

Note

Although Wittgenstein stalwartly refuses to give examples of atomic facts, let's suppose, for the sake of definiteness that the propositional letters P and Q stand for "Socrates is a philosopher" and "Socrates asks a question", then the above logical fact has as an instance:

> Socrates is a philosopher *if and only if* he asks a question is *tautologically equivalent* to it is *not* the case that Socrates is *both* a philosopher and asks a question *only if* it is the case that Socrates is *neither* a philosopher *nor* asks a question.

We can construct a truth table for this "molecular fact":

P	Q	(P	↔	Q)	↔	[~	(P	∧	Q)	→	~	(P	∨	Q)]
T	T	T	T	T	T	F	T	T	T	T	F	T	T	T
T	F	T	F	F	T	T	T	F	F	F	F	T	T	F
F	T	F	F	T	T	T	F	F	T	F	F	F	T	T
F	F	F	T	F	T	T	F	F	F	T	T	F	F	F
		1	3	2	9	6	1	4	2	8	7	1	5	2

The initial two columns on the left assign to the component propositions P and Q all the possible truth assignments. These correspond to the four logically possible worlds—the world in which both P and Q are true, the world in which P is true but Q is false, the world in which P is false but Q is true, and the world in which both P and Q are false. Filling out the truth table is a matter of computing the various columns based on the truth rules for each of the connectives:

- The *negation* $\neg P$ has the *opposite* truth-value of P.
- A *conjunction* $(P \wedge Q)$ is *true* if and only if *both conjuncts* P and Q are *true*.
- A *disjunction* $(P \vee Q)$ of sentences is *false* if and only if *both disjuncts* P and Q are *false*.
- A *conditional* $(P \rightarrow Q)$ is *false* if and only if its *antecedent* P is *true* and its *consequent* Q is *false*.
- A *biconditional* $(P \leftrightarrow Q)$ is true if and only if its *constituents* P and Q have the *same truth value*.

	1.1	The world is the totality of facts, not things.
	1.11	For the totality of facts determines both was what is the case, and also whatever is not the case.
	1.12	The facts in logical space are the world.
	1.13	The facts in logical space are the world.
	1.2	The world divides into facts.
2		What is the case—a fact—is the existence of states of affairs.
	2.01	A state of affairs (a state of things) is a combination of objects (things).

Picture Theory of Meaning: Pictures (2.1 – 2.225); Thoughts (3 – 3.13); Propositions and Names (3.14 – 3.261)

1		The world is all that is the case.
	1.1	The world is the totality of facts, not things.
	1.11	For the totality of facts determines both was what is the case, and also whatever is not the case.
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Wittgenstein's theory of factual propositions depends on the fundament idea of exclusion. A factual proposition always excludes certain possibilities. The proposition that P excludes the possibility that not-P. All factual propositions are truth-functions of elementary propositions.

1.1 *Philosophy of Logic*: Convention (3.62 – 3.5), Philosophy (4 – 4.0031), True and False (4.01 – 4.0641); Science (4.1 – 4.116), Formal Concepts (4.12 – 4.2), Truth function (4.21 – 4.45); Tautology (4.46 – 5.101); Inference (5.11 – 5.156)

Wittgenstein's view is that necessity is logical necessity and that the necessary truths of logic are all empty tautologies. This view is tantamount to a denial of Kant's claim there are synthetic a priori truths.

This particular molecular proposition has several interesting logical properties. First, it combines the five most common logical connective and operators $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$ to one a single propositional. Secondly, this proposition is a tautology or logical truth as can be seen from that fact that column 9, the final column to be calculated, has all T's or the truth value true in every row of its final column. Intuitively, this means that the proposition is true in all logically possible worlds. Thirdly, the proposition implicitly contains the two, and only two, logical connectives that are adequate to express all 16 of the possible binary connectives. The symbolic sentence $\neg(P \wedge Q)$, which states that *not both* P and Q are true, is symbolized by the Sheffer stroke $P \mid Q$ and is known as *nand*. The sentences $\neg(P \vee Q)$ which states that neither P nor Q is true, is symbolized by the dagger $P \downarrow Q$ and is known as *nor*. Notice that this last fact is a fact about logic—a *metalogical* fact—of the sort that violates Wittgenstein's dichotomy that truths of logic cannot be said but only shown. Nevertheless, it remains an interesting fact about the logic of logic. Wittgenstein's dichotomy of *showing* and *saying* therefore begs the question against the very idea of meta-mathematics.

1		The world is all that is the case.
	1.1	The world is the totality of facts, not things.
	1.11	For the totality of facts determines both was what is the case, and also whatever is not the case.
	1.12	The facts in logical space are the world.
	1.13	The facts in logical space are the world.
	1.2	The world divides into facts.
2		What is the case—a fact—is the existence of states of affairs.
	2.01	A state of affairs (a state of things) is a combination of objects (things).

Another doctrine of Wittgenstein's was that tautologies are empty and say the same thing, that is nothing at all. From a semantic point of view, all tautologies are true in every logically possible world and hence do not say something distinctive about any possible world. However, not all tautologies are created equal. From a syntactic point of view, some tautologies have more logical depth than others insofar as they have more deductive consequences. The following propositional counterparts to Aristotle's Three Laws of Thought, are tautologies that "say" different things:

Law of Excluded Middle: $\phi \vee \neg\phi$
Law of Non-Contradiction: $\neg(\phi \wedge \neg\phi)$
Law of Identity: $\phi \iff \phi$

And perhaps it is not immediately obvious what the following tautologies "say":

Pierce's Law: $[(P \to Q) \to P] \to P$
Consequentia Mirabilis: $(\neg P \to P) \iff P$
Scotus's Law: $(P \wedge \neg P) \to X$
$(P \to Q) \vee (Q \to R)$

The following set of tautological schemata form an axiomatic basis for conditional logic together with the single rule of inference *modus ponens*:

$(\mathbf{L}_1)$ $\phi \to (\psi \to \phi)$
$(\mathbf{L}_2)$ $[\phi \to (\psi \to \chi)] \to [(\phi \to \psi) \to (\phi \to \chi)]$
$(\mathbf{L}_3)$ $(\neg\psi \to \neg\phi) \to (\phi \to \psi)$

Modus ponens: From $\phi \to \psi$ and ϕ to infer ψ.

1.2 *Syntax of Arithmetic*:

1		The world is all that is the case.
	1.1	The world is the totality of facts, not things.
	1.11	For the totality of facts determines both was what is the case, and also whatever is not the case.
	1.12	The facts in logical space are the world.
	1.13	The facts in logical space are the world.
	1.2	The world divides into facts.
2		What is the case---a fact---is the existence of states of affairs.
	2.01	A state of affairs (a state of things) is a combination of objects (things).

Wittgenstein's view that numbers are the exponents of functions, while suggestive of the successor operation, does not give a rigorous analysis of numbers. Instead, Wittgenstein's manipulation of exponents seems to presuppose, rather than provide an explanation for, the laws of arithmetic.

1.3 *Solipsism*: Belief (5.54 – 5.5423), Self and World (5.55 – 5.641)

1		The world is all that is the case.
	1.1	The world is the totality of facts, not things.
	1.11	For the totality of facts determines both was what is the case, and also whatever is not the case.
	1.12	The facts in logical space are the world.
	1.13	The facts in logical space are the world.
	1.2	The world divides into facts.
2		What is the case---a fact---is the existence of states of affairs.
	2.01	A state of affairs (a state of things) is a combination of objects (things).

The problem with solipsism is that when the solipsist denies the existence of anything but himself, he uis unable to point to what it is that, according to him, does not exist.

1.4 *Logic, Science, and Mathematics*: Logic and Mathematics (6 – 6.241), Science (6. – 6.372)

1		The world is all that is the case.
	1.1	The world is the totality of facts, not things.
	1.11	For the totality of facts determines both was what is the case, and also whatever is not the case.
	1.12	The facts in logical space are the world.
	1.13	The facts in logical space are the world.
	1.2	The world divides into facts.
2		What is the case---a fact---is the existence of states of affairs.
	2.01	A state of affairs (a state of things) is a combination of objects (things).

Wittgenstein believed himself "to have found, on all essential points, the final solution to the problems' of philosophy." The goal is to deal 'with the problems of philosophy' by 'set[ting] a limit thought.' The way to do this is to set a limit 'not to thought, but to the expressions of thoughts,' i.e., to language. Wittgenstein's principles are too restrictive:

- Wittgenstein's Principle of ________________ (2.02) _____________ physics.
- Wittgenstein's Principle of ________________ (4.2211) __________ mathematics.
- Wittgenstein's Principle of _________________(5.54) ____________ psychology.

1.5 *Value and Mysticism*: Value (6.373 – 6.522), *What cannot be said* (6.53 – 7)

1		The world is all that is the case.
	1.1	The world is the totality of facts, not things.
	1.11	For the totality of facts determines both was what is the case, and also whatever is not the case.
	1.12	The facts in logical space are the world.
	1.13	The facts in logical space are the world.

	1.2	The world divides into facts.
2		What is the case---a fact---is the existence of states of affairs.
	2.01	A state of affairs (a state of things) is a combination of objects (things).

The reason why the problems of philosophy ('metaphysical problems in particular') 'are posed is that the logic of our language is misunderstood.' Therefore, what 'can be said at all can be said clearly, and what we cannot talk about we must consign to silence.'

Wittgenstein's famous ladder passage is anticipated, as pointed out by Hao Wang, in the Chinese Diamond Sutra ("The *dharma* I am preaching is analogous to a raft [which is to be discarded after use]' even the *dharma* can be discarded, *a fortiori* the non-*dharma*". It is also anticipated by Sextus Empiricus' *Against the Logicians*, II, 480-1 ("And again, just as it is not impossible for a man who ascends to a high place to overturn the ladder with his foot after the ascent, so also it is not unlikely that the Skeptic after he has arrived at the demonstration of his thesis by means of the argument…as it were a step ladder, should then abolish this very argument").

Wittgenstein did not follow his own advice in proposition 7. So perhaps it is fitting to conclude with the paradoxical humor of Zhūangzi:

> 荃者所以在魚，得魚而忘荃；蹄者所以在兔，得兔而忘蹄；言者所以在意，得意而忘言。吾安得夫忘言之人而與之言哉！
>
> *The bamboo fish net exists for catching fish. Once the fish is caught, forget the net! The rabbit snare exists for trapping rabbits. Once the rabbit is trapped, forget the snare! Words exist because they are used for expressing meaning. Once the meaning is grasped, forget the words! Where can I meet those who have forgotten words so I can have a word with them?*

2 ‘Must’ and ‘Might’ — The Modal Logic of Necessity and Possibility

2.1 Modes of Truth and Modal Logics

Historically, notions like *necessity*, *possibility*, *impossibility*, and *contingency* were thought of as modes of truth or ways in which a proposition could be true or false. *Modal logic* began as the study of the logic of these modes of truth.

Aristotle, in Chapter 9 of *De Interpretatione*, discusses modality in his famous example of the sea battle. Suppose the sea battle will be fought tomorrow. Then it was true yesterday that it would be fought tomorrow. So if all past truths are necessarily true, then it is necessarily true now that the battle will be fought tomorrow. A similar argument holds on the supposition that the sea battle will not be fought tomorrow. Aristotle proposed solving this problem of *logical fatalism* by denying that future contingent propositions have definite truth-values.

Using the ‘$\Box$’ for ‘it is necessary that’, the principle that all necessary truths are in fact

$$\Box P \rightarrow P, \qquad \textbf{(T)}$$

but adding its converse that all truths are necessary truths:

$$P \rightarrow \Box P \qquad \textbf{(V)}$$

collapses the notions of truth and necessary truth.

Medieval philosophers, concerned with such theological issues as articulating the nature of the Trinity, appealed to such modal notions as essence and accident, contingency and necessity in their labyrinthine theological reflections.[3] Akin to the problem is logical fatalism is the problem of *theological fatalism*: the problem of reconciling divine foreknowledge and human freedom. Saint Augustine (354–430) in his treatise *On the Free Choice of Will* considers an argument for *theological fatalism* proposed by Evodius. Evodius argued that “God foreknew that man would sin, that which God foreknew must necessarily come to pass.” We may set forth this argument for theological fatalism for a particular case as follows:

> If God knew that Adam would sin, then, necessarily Adam sinned.
> God knew that Adam would sin (because God is omniscient).
> Therefore, Adam necessarily sinned.

St. Thomas Aquinas (1225–1274) in his *Summa Contra Gentiles* (part I, chapter 67) criticized this kind of argument as resting on an amphiboly. The critical first premise “if God knew Adam would sin, then, *necessarily*, Adam sinned” is ambiguous between

> (1*a*) It is necessarily the case that if God knows that Adam will sin then Adam will sin.
> (1*b*) If God knows that Adam will sin, then it is necessary that Adam will sin.

Aquinas called (1*a*) the *necessity of the consequence* contrasting it with (1*b*) the *necessity of the consequent*. Using the ‘$\Box$’ to abbreviate ‘it is necessary that’, the difference between these two can be made more perspicuous with symbols:

$$\Box(P \rightarrow Q) \qquad (1a)$$

$$(P \rightarrow \Box Q) \qquad (1b)$$

[3] The theological, if not the political, roots Great Schism of 1054, which can be traced back to a disagreement about the modalities of the Persons of the Trinity. The Nicean Creed (325) uses the term *homoousios* (from the Greek *homo* = ‘same‘ and *ousios* = ‘essence’ or ‘substance’), in contrast to *homoiousios* (from the Greek *homoi* = ‘similar’) making the solitary *i* the jot and tittle of Nicean Creedal Orthodoxy. The Greek Church preferred the latter term since the former had been used by the Syrian Bishop of Antioch to espouse *modal monarchism*, the heresy that the Heavenly Father, Resurrected Son and Holy Spirit are not three distinct Persons, but are rather different *modes* or *aspects* of one monadic God perceived by believers as distinct persons. The Latin Church adopted the former siding with Athanasius against the heretic Arius, who denied that Jesus was co-equal and co-eternal with the Father. The Second Council of Nicea (381), among other changes, inserted the word *filioque* (from the Latin *filio* = the son, and *que* = “and”) into the Nicean-Constantinopolitan Creed and the Latin mass. In Latin theology, the three Persons of the Trinity are logically distinguished by the formal relations of “proceeding from” citing such proof texts as Phil. 1:9, Titus 3:6, Acts 2:33. Orthodox theology, citing the words of Jesus in proof texts such John 15:26, regarded the insertion of *filioque* into the Nicean-Constantipolitan creed, is as Semi-Sabellianism. The great church historian Pelikan (1988: 90) opined: “If there is a special circle of the inferno described by Dante reserved for historians of theology, the principal homework assigned to that subdivision of Hell for at least the first several eons of eternity may well be a thorough study of all the treatises… devoted to the inquiry: Does the Holy Spirit proceed from the Father only, as Eastern Christendom contends, or from both the Father and the Son as the Latin Church teaches?” In 1989 Pope John Paul II and Patriarch Demetrius knelt together in Rome and recited the Nicene Creed

Solving the famous theological problem of reconciling divine foreknowledge with human freedom may turn on exposing ambiguities of this sort.

Perhaps the most famous theological application of modal logic is Saint Anselm's modal ontological argument. According to Saint Anselm (1033–1109), it follows from God's nature that it is necessary that God exists if God exists at all. Moreover, this conditional itself, being a conceptual truth, is itself necessarily true. We then have the following argument:

> Necessarily, if God exists, then God necessarily exists.
> It is possible that God exists.
> Therefore, God (actually) exists.

Using '$\Diamond$' for 'it is possible that', the above argument can be symbolized:

$$\Box(G \rightarrow \Box G). \quad \Diamond G \quad \therefore G \tag{2}$$

The question of whether Anselm's argument is valid became a precise question when various systems of modal logic were proposed and developed in the 1960s.

Gottlob Frege (1848–1925), the inventor (or discoverer) of modern predicate-quantifier logic, relegated modality to autobiographical information about the speaker, and for many years logicians only investigated extensional logic.

One of the most puzzling validities, at least to the beginning logical students is known as *Lewis's Dilemma*:

$$P \wedge \neg P \rightarrow Q,$$

which states *"a contradiction implies anything"*[4]. This implication follows from the inference rules of simplification, addition, and *modus tollendo ponens*[5], which are themselves not particularly puzzling.

[4] Or *"ex falso quodlibet"*, in Latin.
[5] *"mode that denies by affirming"*

1.	~~*Show*~~	$P \wedge \neg P \rightarrow Q$	6, CD
2.		$P \wedge \neg P$	Assume (CD)
3.		P	2, S
4.		$\neg P$	2, S
5.		$P \vee Q$	3, ADD
6.		Q	5, 4 MTP

The following theorems are known as the *paradoxes of material implication*:

$$\neg P \rightarrow (P \rightarrow Q) \tag{T18}$$

← law of denying the antecedent

$$Q \rightarrow (P \rightarrow Q) \tag{T2}$$

← law of affirming the consequent

$$(P \rightarrow Q) \vee (Q \rightarrow \neg Q) \tag{T58}$$

← conditional excluded middle

$$(\neg P \rightarrow P) \rightarrow P \tag{T114}$$

← Consequentia Mirabilis, "admirable consequence" [Cantor's Δ?]

C. I. Lewis investigated modal logic in order to find a stricter form of the conditional which would not result in such paradoxes. Lewis defined *strict implication* $P \Rightarrow Q$ (read "P *strictly implies* Q") by combining modality with the truth-functional conditional:

$$P \Rightarrow Q \quad := \quad \Box(P \wedge \neg Q),$$

or alternatively,

$$P \Rightarrow Q \quad := \quad \Box(P \rightarrow Q).$$

The notion of strict implication was characterized by such axioms as:

The philosopher Leibniz (1646–1716) explicitly invoked that language of possible worlds to explain the difference between necessary and contingent truths. What is logically or necessarily true are those truths truth in *all* possible worlds, whereas a contingent truth is one that is true in *some* possible worlds.

Drawing upon this logical connection between universal and existential quantification and the modal notions of necessity and possibility, we obtain a modal version of the classical Aristotelian Square of Opposition and the duality of modal laws known as the laws of modal negation.

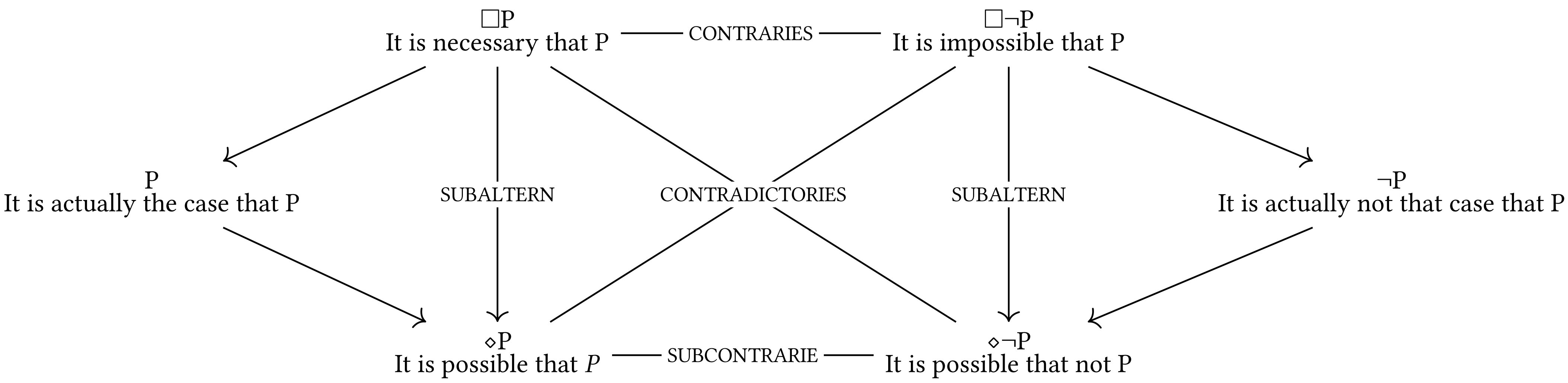


Figure 2.1: An Aristotelian Diamond of Opposition

In the modern development of modal logic, logicians noticed that a host other phenomena—such as deontic notions of obligation and permissibility, epistemic notions of knowledge and belief, as well as temporal operators—share these logical relations and hence can be represented as modal logics.

In *deontic logic*, □ is read "it is morally obligatory that" and ⊠ is read "it is morally permissible that". Kant's maxim that "ought implies can", that is, whatever is obligatory is permissible, is captured by modal axiom **(D)**:

$$\Box P \rightarrow \boxtimes P. \quad \textbf{(D)}$$

In *epistemic logic*, □ is read for some subject *S* "it is known that" and ⊠ is read "it is believed that". Some modal axioms for epistemic logic that have been considered include:

→ logical omniscience

$$\Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q) \quad \textbf{(K)}$$

→ law of affirming the consequent

$$\Box P \rightarrow P \quad \textbf{(T)}$$

→ law of affirming the consequent

$$\Box P \rightarrow \Box\Box P \quad \textbf{(4)}$$

→ law of affirming the consequent

$$\neg\Box P \rightarrow \Box\neg\Box P \quad \textbf{(E)}$$

The axiom **(K)** expresses logical omniscience insofar as this axiom requires that the knowledge of an agent is closed under *modus ponens*; and hence such knowers know all the logical consequences of their knowledge.

Axiom **(T)** states the truism that whatever is known is true. Notice that this axiom would be too strong for deontic logic insofar as an action's being obligatory doesn't imply that the agent actually performs that action.

Axiom **(4)** expresses a high degree of *positive introspective knowledge*: if someone knows P, then she knows that she knows that P. Axiom **(E)** on the other hand, expresses a high degree of *negative* introspective knowledge: if someone doesn't know that P, then he knows he doesn't know P. This axiom is contrary to the experience of Socrates: as the gadfly of Athens, Socrates found through his questioning that many of his fellow Athenians did not know what they were talking about but also didn't know that they didn't know. The gadfly of Athens believed his vocation was to sting his fellow Athenians into the awareness they were own ignorance, a service for which they did not always show adequate appreciation.

The *temporal logic* or Diodorian temporal logic was studied by the logician A. N. Prior (1914–1969). To model temporal language, we introduce a pair of modal operators for the future and a pair of modal operators for the past.

$\Box$	It *is always going* [i.e., in *all* futures] to be the case that
$\boxtimes$	It *will>* [i.e., in *some* future] be the case that
$\Box$	It *has always been* [i.e., in *all* pasts] the case that
$\boxtimes$	It *was once*] [i.e., in *some* past or *"once upon a time"*] the case that

The axioms for minimal tense logic include version of Axiom **(K)** for the two necessity operators:

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \quad \text{Whatever has always followed from what always has been, always has been.} \quad (\mathrm{K}\Box)$$

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \quad \text{Whatever will always follow from what always will be, always will be.} \quad (\mathrm{K}\Box)$$

It also contains two axioms concerning the interaction of the past and future that has the form of the so-called Brouwersche axiom with alternating valences:

$$\varphi \to \Box\boxtimes\varphi \quad \text{What is, will always have been} \quad (\mathrm{B}\Box\boxtimes)$$

$$\varphi \to \Box\boxtimes\varphi \quad \text{What is, has always been going to happen} \quad (\mathrm{B}\Box\boxtimes)$$

The Brouwersche **(B)** axiom was so-named by the logician Oskar Becker (Becker 1930) after the charismatic Dutch mathematician L. E. J. Brouwer (1881–1966), who championed a philosophy of mathematics known as *intuitionism*. It happens that when the $\boxtimes$ can be paraphrased as $\boxtimes\neg\boxtimes$, the resulting axiom has the form of the acceptable form of double negation in intuitionistic logic:

Becker, Oskar. 1930. Zur logik der modalitäten.

$$\varphi \to \neg\boxtimes\neg\boxtimes\varphi \quad \mathbf{(B)}$$

According to intuitionism, mathematical objects do not exist as eternal Platonic objects but are constructions in intuition. Intuitionists read the propositional connectives as involving not merely truth, but proof, and so they rejected such classical forms of reasoning as *reductio ad absurdum* and theorems such as the *law of excluded middle.* Intuitionists reject the following mathematical proof.

Around the 1970s it was noticed that the famous incompleteness theorems (Gödel 1931) were propositional in character and that their logic could be captured in propositional modal logic. These modal provability logics added to Axiom **(K)** the following axiom, known as the Gödel-Löb axiom or also the well-ordering axiom.

Gödel, Kurt. 1931. Über formal unentscheidbare sätze der principia mathematica und verwandter systeme i. *Monatshefte für Mathematik Und Physik* 38. 173–198.

$$\Box(\Box\varphi \to \varphi) \to \Box\varphi \quad \mathbf{(W)}$$

Modal Provability Logics proliferated from the 1950s–1970s, but the genesis of the idea goes back to a short note of Gödel (1933b) in which he noted that intuitionistic truth is defined in terms of proof since provability is a kind of necessity. The above axiom can be read as a kind of soundness theorem.

if it is provable that φ being provable implies it is true, then φ is provable.

Gödel, Kurt. 1933b. The present situation in the foundations of mathematics. *Collected Works* 3. 45–53.

A Classical Theorem, but in Intuitionism

Theorem. There exist two irrational numbers x and y such that x^y is irrational.

Proof. Consider

$$\sqrt{2}^{\left(\sqrt{2}^{\sqrt{2}}\right)} = \sqrt{2}^{(\sqrt{2}\cdot\sqrt{2})} = \sqrt{2}^{2} = 2,$$

which is rational.
The number $\sqrt{2}^{\sqrt{2}}$ is either rational or irrational.

- If it's rational, then $x = y = \sqrt{2}$ are both irrational yet x^y is rational.
- If $\sqrt{2}^{\sqrt{2}}$ is irrational, then $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$ are both irrational and x^y is rational.

Either way, there exists x and y such that x^y is rational. $\Box$

Intuitionists reject this classical argument by separation of cases because it does not actually construct the numbers x and y such that

xy is irrational. The idea behind intuitionistic logic is that the connectives are reinterpreted as involving a kind of provability.

In case you're curious, it is actually known that $\sqrt{2}^{\sqrt{2}}$ is irrational, due to the Gelfond–Schneider theorem which verifies that it usually takes significantly more effort to convince an intuitionistic mathematicians than a classical one.

Later we will show how to use a modal provability logic to exhibit the propositional logic of key parts of Gödel's First and Second Incompleteness Theorems.

In contemporary logic, modal logic has grown beyond these philosophical origins and is at the interface of a number of disciplines including the studies information flow and dynamics, game theory, and computability.

Exercises

(1) Symbolize the following modal arguments.

  (A) Eratosthenes must either be in Syene or Alexandria. Eratosthenes cannot be in Syene. Therefore, Eratosthenes must be in Alexandria.

  (B) Assume that justice can be defined as paying your debts and telling the truth. Then it is *morally obligatory* for Cephalus to comply to a madman's request that Cephalus return a borrowed sword and that Cephalus tell the truth about the whereabouts of a friend whom the madman wants to kill with the sword. However, if this act is *morally obligatory*, then it is *morally permissible*. However, it is *morally impermissible* (or *morally forbidden*) for Cephalus to comply. So it isn't *morally obligatory* for Cephalus to comply. Justice, therefore, cannot be defined as paying your debts and telling the truth.

  (C) It is conceivable that I am having experiences qualitatively identical to those I am having now on the supposition that I am being deceived by an evil genius. If that is conceivable, then I do not have indubitable knowledge that the external world exists.

(2) Johan van Benthem (Van Benthem 2010: 12) was asked to symbolize the philosophical claim that "nothing is absolutely relative". He came up with the following:

Van Benthem, Johan. 2010. *Modal logic for open minds.* Stanford, CA: Centre for the Study of Language & Information.

$$\neg\Box(\Diamond\varphi \wedge \Diamond\neg\varphi).$$

Use familiar equivalences from propositional logic and the modal negation laws to show that this symbolization is equivalent to

→ *McKinsey's Axiom*

$$\Box\Diamond\varphi \rightarrow \Diamond\Box\varphi \qquad \textbf{(M)}$$

(3) Match the following symbolizations with the best corresponding translation below.

symbolization	translation
$\Diamond\Box$P	*It was always the case that it will sometime be the case that* P.
$\Diamond$P → $\Diamond$P	*It will sometime be the case that it was once the case that* P.
$\Box\Diamond$P	*Whatever will always be, will be.*
$\Diamond\Diamond$P	*Once upon a time, it was always the case that* P.

2.2

2.3

2.4

2.5

2.6

2.7 Possible World Semantics

The Leibnizian idea of characterizing necessity and possibility in terms of truth in all or some possible worlds was given an elegant formalization by (Kripke 1959; 1963) when he was only a teenager. According to Leibniz, a sentence is *necessary* if it is true in *every* possible world, and a sentence is *possible* if it is true is *some* possible world. Kripke showed that by placing very natural conditions on a relation of *relative possibility* or *accessibility* on a set of possible worlds, the various systems of modal logic could be validated.

Kripke, Saul. 1959. A completeness theorem in modal logic. *Journal of Symbolic Logic* 24(1). 1–14. DOI: https://doi.org/10.2307/2964568
Kripke, Saul. 1963. Semantical analysis of modal logic I normal modal propositional calculi. *Mathematical Logic Quarterly* 9(5-6). 67–96. DOI: https://doi.org/10.1002/malq.19630090502

Intuitively, a possible world tells us for each sentence letter whether it is true or false in that world. Stripping away inessentials, we can represent a possible world by a subset of sentence letters. A *modal structure* $\mathbb{M}$ is an ordered triple $\langle W, R, \alpha\rangle$, where W is a set[6] of possible worlds, $R \subseteq W \times W$ is a relation known as the *accessibility relation* or the *relative possibility relation*, and α is a distinguished element of W known as the *actual world*.

[6] set-theoretic size issues‽

We can exhaustively characterize the notion of

$$\beta \vDash \varphi,$$

the *truth of a sentence in a possible world* β, by

(4) If φ is a sentence letter S, then
$$\beta \vDash S \iff S \in \beta,$$
i.e. S is a member of β;

(5) If φ is ψ, then
$$\beta \vDash \neg\psi \iff \beta \nvDash \psi,$$
i.e., it is not the case that $\beta \vDash \psi$;

(6) (a) If φ is $(\psi \wedge \chi)$, then
$$\beta \vDash (\psi \wedge \chi) \iff \beta \vDash \psi \;\&\; \beta \vDash \chi,$$
i.e., both $\beta \vDash \psi$ and $\beta \vDash \chi$;

(b) If φ is $(\psi \vee \chi)$, then
$$\beta \vDash (\psi \vee \chi) \iff \beta \vDash \psi \parallel \beta \vDash \chi,$$
i.e., either $\beta \vDash \psi$ or $\beta \vDash \chi$, or both;

(c) If φ is $(\psi \to \chi)$, then
$$\beta \vDash (\psi \to \chi) \iff \beta \vDash \psi \implies \chi,$$
i.e., if $\beta \vDash \psi$ then $\beta \vDash \chi$, or either $\beta \nvDash \psi$ or $\beta \vDash \chi$ ‽;

(d) If φ is $(\psi \leftrightarrow \chi)$, then
$$\beta \vDash (\psi \leftrightarrow \chi) \iff \beta \vDash \psi \iff \chi,$$
i.e., $\beta \vDash \psi$ if and only if $\beta \vDash \chi$;

Finally, the law clause gives the Leibnizian truth conditions for necessity and possibility:

(7) (a) If φ is $\Box\psi$, then
$$\beta \vDash \Box\psi \iff \forall\gamma \in W.\, \beta\, R\, \gamma \implies \beta \vDash \psi,$$
i.e., ψ is true in *all* possible worlds γ, possible-relative to β;

(b) If φ is $\Diamond\psi$, then
$$\beta \vDash \Diamond\psi \iff \exists\gamma \in W.\, \beta\, R\, \gamma \implies \beta \vDash \psi,$$
i.e., ψ is true in some possible worlds γ, possible-relative to β;

This completes the definition of truth in a model for modal propositional logic. Using this definition of truth, we can now define what it means for a sentence to be *semantically valid*:

$$\vDash \varphi \text{ (i.e., } \varphi \text{ is semantically valid)} \iff \forall \alpha \in W.\, \alpha \vDash \varphi.$$

Next, we obtain different systems of modal logic when various conditions are placed on the accessibility or relative possibility relation R. We say that a relation R is a *series* if R is serial; R is a *reflexivity* if R is totally reflexive; R is a *similarity* if R is totally reflexive and symmetric; R is a *partial ordering* if R is totally reflexive and transitive; R is an *equivalence relation* if R is totally reflexive and euclidean. It turns out that the axioms of modal logic discussed above are validated when natural conditions are imposed on the accessibility or relative possibility relation R.

Table 2.3: Properties of Accessibility

D	$\Box\varphi \to \Diamond\varphi$	serial	$\forall\alpha.\exists\beta.\alpha\,R\,\beta$
T	$\Box\varphi \to \varphi$	reflexive	$\forall\alpha.\alpha\,R\,\alpha$
B	$\varphi \to \Box\Diamond\varphi$	symmetric	$\forall\alpha.\forall\beta.(\alpha\,R\,\beta \Rightarrow \beta\,R\,\alpha)$
4	$\Box\varphi \to \Box\Box\varphi$	transitive	$\forall\alpha\forall\beta.\forall\gamma.(\alpha\,R\,\beta\ \&\ \beta\,R\,\gamma \Rightarrow \alpha\,R\,\gamma)$
5	$\Diamond\varphi \to \Box\Diamond\varphi$	euclidean	$\forall\alpha.\forall\beta.\forall\gamma.(\alpha\,R\,\beta\ \&\ \alpha\,R\,\gamma \Rightarrow \beta\,R\,\gamma)$

Systems of modal logic are *normal* when everything derivable from necessary truths is itself necessary. This will be the case if the rule of *modus ponens* and axiom (K) are valid:

$$\Box(P \to Q) \to (\Box P \to \Box Q). \qquad \textbf{(K)}$$

Axiom **K** expresses the intuition that *necessary truths imply only necessary truths.*

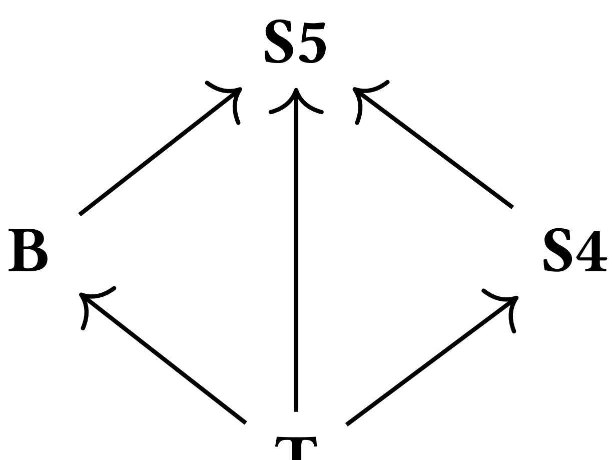
We can conveniently summarize the above systems of modal logic in a chart. The various modal systems can be characterized by the axioms that are valid in them. The smallest normal modal logic system **K** contains axiom **K**. The four most famous modal logics are system (T) named after the Gödel–Feyes–von Wright modal logic to model tautologies, system **S4** and **S5**, named after C. I. Lewis's axioms for strict implications, and the unduly neglected Brouwersche system **B**, named by Becker after the intuitionist L. E. J. Brouwer due to the characteristic axiom's similarity to intuitionistic double negation. All these systems contain **K** and **T**. The system **D** is a weaker system than **T**, containing axioms **K** and **D**, is named for deontic logic.

Notice that relationships of containment among the modal systems follow from the logic of relations. Axiom **B** requires that R be symmetric and **4** requires that R be transitive. System **S5** with axioms **T** and **E** require R to be an equivalence relation (i.e., reflexive, symmetric, and transitive); hence, **S5** could also be specified by requiring axioms **T**, **4**, and **B** to be valid. Therefore, **S5** contains **S4** and **B**, neither of which contains the other. Systems **S5**, **S4** and **B** all contain system **T**, which contains **D**.

A convenient way of describing these modal logics is by their *Lemmon code* listing the axioms valid in them. For example, **S5** = **KTE** = **KT4B** = **KD4B**. We can represent these containment relations in a diagram in which downward paths represent containment.

Table 2.4: Modal Axioms and Accessibility

D	Deontic Logic	**D**	$\Box P \to \Diamond P$	**KD**	seriality
T	Gödel-Feys-von Wright Tautology	**T**	$\Box P \to P$	**KT**	reflexivity
B	Brouwersche System	**B**	$P \to \Box\Diamond P$	**KTB**	similarity
S4	Lewis' Strict Implication System 4	**4**	$\Box P \to \Box\Box P$	**KT4**	partial ordering



S5	Lewis' Strict Implication System 5	**E**	$\Diamond P \rightarrow \Box\Diamond P$	**KTE**	equivalence relation

Various deontic modal systems can also be characterized by their axioms (*Lemmon code*).

Table 2.5: Lemmon Codes for Deontic Modal Systems

System **T**	**KT**	Deontic **D**	**KD**
System **B**	**KTB**		
System **S4**	**KT4**	Deontic **S4**	**KD4**
System **S5**	**KTE** = **KT4B** = **KD4B**		

The logical relationships among the above systems of modal logic can be set forth in a diagram (due to Krister Segerberg who omits **KD5** and **K45**). As before, a modal logic is included in another if it is connected to it, directly or indirectly, by an upward path. The second diagram is a more elaborate Picassos' Electric Chair that includes deontic modal logics.

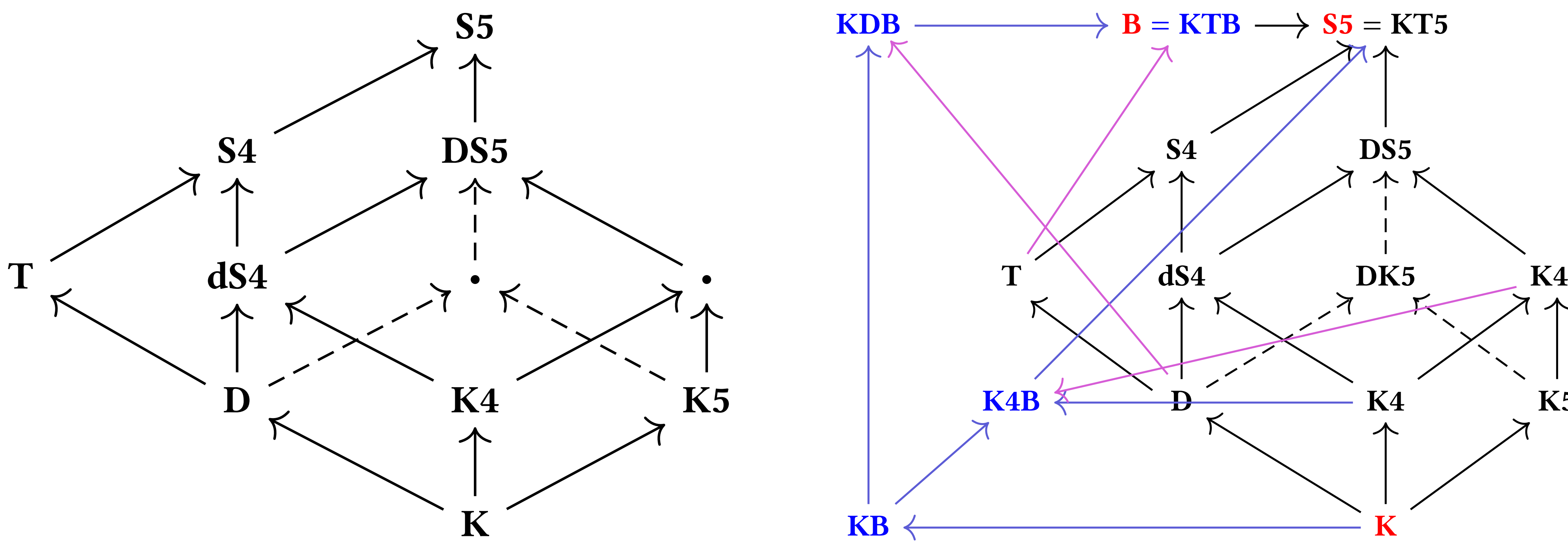


Figure 2.3: Hasse–Picassos Diagrams for Systems of Modal Logic

One way to visualize how the conditions on the accessibility relation validate their respective axioms is using the definitions of '□' and ◇ in terms of possible worlds and use directed graphs from Chapter IV to represent the accessibility relation R. Here the accessibility relation $\alpha\, R\, \beta$ (read "β is possible relative to α" " or "β is accessible to α") is represented by an arrow from a circle representing possible world α to a circle representing possible world β.

We can, using the directed graphs from the theory of relations, translate properties of accessibility relations into geometric properties of directed graphs. Symmetry, for example, requires that all accessibility arrows are double arrows. Reflexivity requires that every world be accessible to itself and so every world has a loop, which is a special case of a double arrow. Seriality requires that every world is a tail of an arrow. Transitivity requires that for every indirect path of accessibility from α to β and from β to γ, there is a direct path from α to γ. Being euclidean and serial is equivalent to being an equivalence

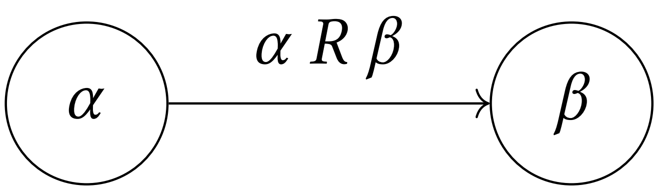


Figure 2.4
Accessibility Represented by Directed Graphs

relation, that is, being reflexive, symmetric and transitive. Expressing R in terms of love and worlds in terms of persons, we have the following intuitive translations:

Table 2.6: Relational Properties of Directed Graphs

serial	$\forall\alpha.\exists\beta.\alpha\ R\ \beta$	Everyone is a lover.
reflexive	$\forall\alpha.\alpha\ R\ \alpha$	Everyone is a self-lover.
symmetric	$\forall\alpha.\forall\beta.(\alpha\ R\ \beta \Rightarrow \beta\ R\ \alpha)$	All love is requited; all love is mutual.
transitive	$\forall\alpha.\forall\beta.\forall\gamma.(\alpha\ R\ \beta\ \&\ \beta\ R\ \gamma \Rightarrow \alpha\ R\ \gamma)$	Love is transitive.
euclidean	$\forall\alpha.\forall\beta.\forall\gamma.(\alpha\ R\ \beta\ \&\ \alpha\ R\ \gamma \Rightarrow \beta\ R\ \gamma)$	All beloveds of the same lover, love each other and themselves.

Using the definition of truth in a modal system set forth above, we can rigorously demonstrate that if R is transitive, then axiom **4** is valid. This demonstration is carried out in the meta-language. We use the symbols '$\forall$', '$\exists$', '&', '$\implies$' and '$\in$' in the meta-language for 'all', 'some', 'and', 'if… then', and 'is an element of', respectively. Once the truth clauses are unpacked, the logical demonstration is no more complicated than a derivation in the theory of relations.

Listing 2.1

Montague-style Semantic Derivation of Transitivity Validating Axiom 4

1.	~~Show~~ R is transitive $\implies\ \vDash(\Box\varphi\to\Box\Box\varphi)$	3, CD
2.	$\forall\alpha\,\forall\beta\,\forall\gamma\,(\alpha\ R\ \beta\ \&\ \beta\ R\ \gamma \implies \alpha\ R\ \gamma)$	Assume (CD), def. transitive
3.	~~Show~~ $\vDash(\Box\varphi\to\Box\Box\varphi)$	24, DD
4.	~~Show~~ $\forall\alpha\in W.(\vDash_\alpha\Box\varphi \implies \vDash_\alpha\Box\Box\varphi)$	5, UD
5.	~~Show~~ $\vDash_\alpha\Box\varphi \implies \vDash_\alpha\Box\Box\varphi$	8, CD
6.	$\vDash_\alpha\Box\varphi$	Assume (CD)
7.	$\forall\beta\in W.(\alpha\ R\ \beta \implies \vDash_\alpha\varphi)$	6, truth-def. $\Box$
8.	~~Show~~ $\vDash_\alpha\Box\Box\varphi$	23, DD
9.	~~Show~~ $\forall\beta\in W.(\alpha\ R\ \beta \implies \vDash_\beta\Box\varphi)$	10, UD
10.	~~Show~~ $\alpha\ R\ \beta \implies \vDash_\beta\Box\varphi$	12, CD
11.	$\alpha\ R\ \beta$	Assume (CD)
12.	~~Show~~ $\vDash_\beta\Box\varphi$	22, DD
13.	~~Show~~ $\forall\gamma\in W.(\beta\ R\ \gamma \implies \vDash_\gamma\varphi)$	14, UD
14.	~~Show~~ $\beta\ R\ \gamma \implies \vDash_\gamma\varphi$	16, CD
15.	$\beta\ R\ \gamma$	Assume CD
16.	~~Show~~ $\vDash_\gamma\varphi$	22, DD
17.	$\alpha\ R\ \beta\ \&\ \beta\ R\ \gamma$	11, 15 ADJ
19.	$\alpha\ R\ \gamma$	18, 17 MP
20.	$\alpha\ R\ \gamma \implies \vDash_\gamma\varphi$	7, UI
21.	$\vDash_\gamma\varphi$	20, 19 MP
22.	$\vDash_\beta\Box\varphi$	12, truth-def. $\Box$
23.	$\vDash_\alpha\Box\Box\varphi$	9, truth-def. $\Box$
24.	$\vdash(\Box\varphi\to\Box\Box\varphi)$	4, def. $\vDash$ (*logical consequence*)

The above semantic derivation can be visualized graphically to prove the above result contrapositively. We will show that if

$$\Box\varphi \rightarrow \Box\Box\varphi$$

is false, then the accessibility relation cannot be transitive. Axiom (4) fails when its antecedent φ is true but its consequent $\Box\varphi$ is false in α. By modal negation, $\neg\Diamond\Diamond\varphi$ is equivalent to $\Diamond\Diamond\neg\varphi$.

So both $\Box\varphi$ and $\Diamond\Diamond\neg\varphi$ are true in some possible world α. Eliminating the first $\Diamond$, we have that in some possible world β accessible to α (i.e. $\alpha\, R\, \beta$), $\Diamond\neg\varphi$ is true. Applying the definition of truth for $\Diamond$ again, we have that for some world γ accessible to β (i.e. $\beta\, R\, \gamma$), $\neg\varphi$ is true in γ. Notice that the transitivity of the accessibility relation R would require that γ be accessible to α, which would contradict that $\Box\varphi$. The invalidity of Axiom (**4**) implies the failure of the transitivity of the accessibility relation R. Stating this result contrapositively, if the accessibility relation R is transitive, then axiom (**4**) is valid.

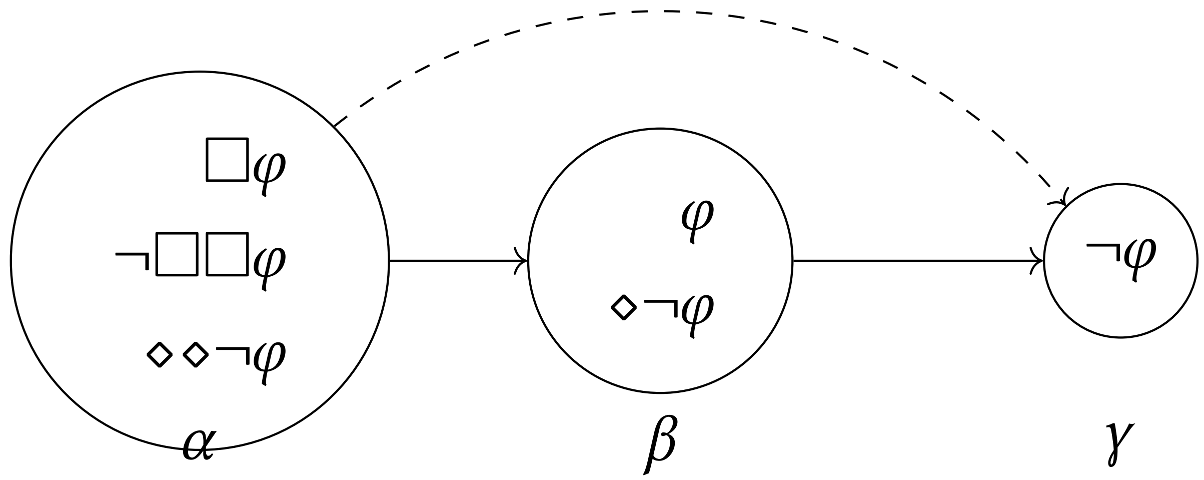


Figure 2.5: Kripke–Mar Diagram for Transitivity of R Validating Axiom (4)

The semantic demonstration together with the graphic proof using directed graphs support one another in building our intuitions for modal logic. The graphic proofs are satisfying because of their simplicity and ability to make the critical step visually perspicuous. On the other hand, the semantic proof is satisfying insofar as the explicit definitions of $\Box$, $\Diamond$, $\vDash$ are shown to work elegantly and precisely with our natural deduction systems using the quantifier logic with UD, UI, EI, EG and the theory of relations. We will use the directed graphs in the context of discovering new axioms and the semantic proofs in the context of justifying that these axioms correspond to imposing requirements on the accessibility relation.

Exercises

TODO... too many graphics... need some rest today...

3 Gödel's Second Incompleteness theorem

Gödel informally explained his First Incompleteness Theorem noting that the analogy to the antinomy of the Liar "leaps to the eye (GCW-I, 149)". Whereas the Liar sentence asserts of itself that it is *untrue*, the Gödel sentence says of itself that it is *unprovable* in a precisely specified formal system such as *Principia Mathematica.*

> ⚠ *Principia Mathematica*:
>
> G : G is not provable in system PM.

Gödel further remarks that *any* epistemological antinomy can be used to motivate his proof but chooses the Richard Paradox, which we discussed in Chapter 1. Gödel's choice was prescient: the Berry Paradox is a simplification of the Richard Paradox, and can be used to elegantly prove the First Incompleteness Theorem showing its connection with Chaitin's interpretation Gödel Incompletehes in terms of algorithmic randomness.

> GÖDEL'S FIRST INCOMPLETENESS THEOREM: if a formal system is consistent and its axiom system has enough arithmetic so that its theorems can be listed by some mechanical procedure, then there exists an *undecidable* sentence in that formal system, which is therefore *incomplete.*[7]

[7] The formal system is also essentially incomplete, i.e., one can add the undecidable Gödel sentence as a new axiom and the resulting system will have a new undecidable sentence, which is also undecidable in the original system.

3.1 Provability Modal Logics

Elegant proofs of Gödel's Second Incompleteness Theorem were discovered in *modal provability logics,* which emerged from the 1950s-1970s. These logics were anticipated by (Gödel 1933a) *"An interpretation of intuitionistic propositional calculus."* Gödel's insight was that intuitionistic *truth* was characterized in terms of *proof*, which is a kind of *necessity*, and so modal axioms could be used to formalize the *properties of provability*:

$$\Box(P \to Q) \to (\Box P \to \Box Q)$$

Gödel, Kurt. 1933a. Eine interpretation des intuitionistischen aussagenkalküls. *Ergebnisse Eines Mathematischen Kolloquiums* 4. 39–40.

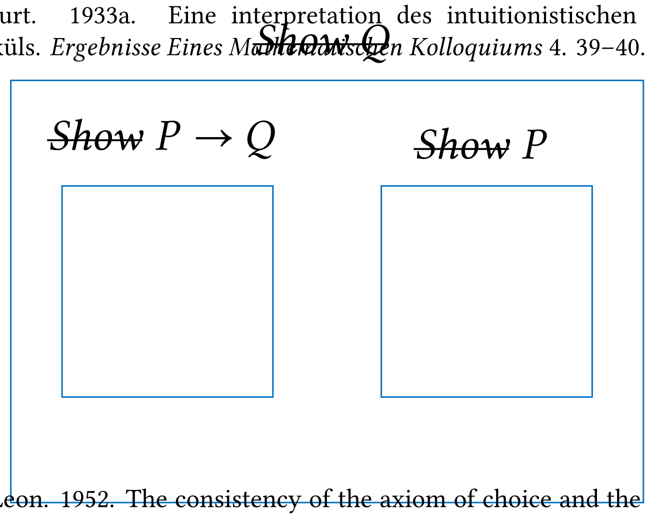


Figure 3.1

Table 3.1: Properties of Provability

(T)	$\Box P \to P$	*What is provable is* ________________.
(K)	$\Box(P \to Q) \to (\Box P \to \Box Q)$	*Whatever follows from what is provable is* ________________.
(4)	$\Box P \to \Box\Box P$	*What is provable is provably* ________.

(Henkin 1952) posed the intriguing question whether the *positive* Gödelian sentence "I am *provable*" is provable. (Löb 1955) answered Henkin's question in the affirmative by showing that Peano Arithmetic proves a counterpart to Löb's Axiom:

Henkin, Leon. 1952. The consistency of the axiom of choice and the generalized continuum-hypothesis with the axioms of set theory. *Journal of Symbolic Logic* 17(3). 207–208. DOI: https://doi.org/10.2307/2267706

Löb, M.H. 1955. Solution of a problem of leon henkin. *The Journal of Symbolic Logic* 20(2). 115–118. Retrieved from http://www.jstor.org/stable/2266895

(L)	$\Box(\Box P \to P) \to \Box P$	*Löb's Axiom restricts (**T**) to what is provable.*

A *Gödel-Löb modal probability logic* (**GL**) results from adding (**L**) to (**K**) and

1. the rule of *necessitation* or *universal derivation* [UD] (*i.e.,* if $\mathbf{GL} \vdash P$ then $\mathbf{GL} \vdash \Box P$),
2. *modus ponens* [MP]
3. a rule for proving all *tautologies* or the KM2 system of natural deduction.

Note: adding axiom (**4**) turns out to be redundant. In 1975 Howard de Jongh proved that Axiom (**4**) is derivable from Löb's axiom and (**K**) using the substitution of '$(\Box P \land P)$' for 'P'.

A *provability system* consists minimal modal logic **K** with the additional axiom

$$(\mathbf{w}) \quad \Box(\Box\varphi \to \varphi) \to \Box\varphi.$$

Theorem 3.1 (Howard de Jongh, 1970s).[8] *Axiom (4) holds in the Gödel-Lob Provability Logic.*

Proof. Replacing φ by $(\Box\varphi \wedge \varphi)$ in Axiom (**W**) we obtain

$$\Box(\Box(\Box\varphi \wedge \varphi) \to (\Box\varphi \wedge \varphi)) \to \Box(\Box\varphi \wedge \varphi).$$

□

It happens that the lemma

$$\Box\varphi \to \Box(\Box(\Box\varphi \wedge \varphi)) \to \Box(\Box\varphi \wedge \varphi)$$

is derivable in minimal modal logic **K** (see exercises below).

It therefore follows that

$$\Box\varphi \to \Box(\Box\varphi \wedge \varphi)$$

and hence by (**T**),

$$\Box\varphi \to \Box\Box\varphi.$$

3.2 Exercises

Listing 3.1

Exercise Lemma

1.	~~Show~~ $\Box\varphi \to \Box(\varphi \to (\Box(\Box\varphi \wedge \varphi) \to (\Box\varphi \wedge \varphi)))$	
2.	$\Box\varphi$	Assume (______)
3.	$\Box(\Box(\Box\varphi \wedge \varphi) \to \Box\varphi \wedge \varphi) \to \Box(\Box\varphi \wedge \varphi)$	Axiom (L)
4.	~~Show~~ $\Box(\varphi \to (\Box(\Box\varphi \wedge \varphi) \to (\Box\varphi \wedge \varphi)))$	5, ______
5.	~~Show~~ $\varphi \to (\Box(\Box\varphi \wedge \varphi) \to (\Box\varphi \wedge \varphi))$	7, ______
6.	φ	Assume (______)
7.	~~Show~~ $\Box(\Box\varphi \wedge \varphi) \to \Box\varphi \wedge \varphi$	11, ______
8.	$\Box(\Box\varphi \wedge \varphi)$	Assume (______)
9.	$\Box\Box\varphi \wedge \Box\varphi$	distribution $\Box / \wedge$
10.	$\Box\varphi$	9, S
11.	$\Box\varphi \wedge \varphi$	10, 6 ADJ

Next comes the Löb-Gödel Theorem, which we shall call the "Magical Modal Mystery Tour"[9]:

Now we may obtain a modal version of Gödel's First Incompleteness Theorem. Gödel's famous arithmetical version of the Liar G intuitively says, "I am not provable":

$$G \leftrightarrow \sim\Box G.$$

Note the consistency of Peano Arithmetic may be formulated as: $\sim\Box(1 = 2)$ (von Neumann) or $\sim\Box(0 \neq 0)$.

[8] *At the Tenth International Tbilisi Symposium on Language, Logic and Computation in 2013, one of the speakers referred to this theorem as part of the folklore of the field not realizing that de Jongh was in the audience.*

[9] van Benthan [2010], p. 245 describes this theorem "as a piece of 'magical' modal reasoning that has delighted generations."

Listing 3.2

Exercise "Löb-Gödel Theorem"

1.	$\vdash \Box\varphi \to \varphi$	Assumption (**T**)
2.	$\vdash \sigma \leftrightarrow (\Box\sigma \to \varphi)$	Gödel's Fixed-Point Theorem
3.	~~*Show*~~ φ	20, ____
4.	~~*Show*~~ $\Box\sigma$	16, ____
5.	~~*Show*~~ $\Box\sigma \to \varphi$	14, ____
6.	$\Box\sigma$	Assume (____)
7.	~~*Show*~~ $\Box\,[\sigma \leftrightarrow (\Box\sigma \to \varphi)]$	8, ____
8.	$\sigma \leftrightarrow (\Box\sigma \to \varphi)$	2, Gödel's Fixed-Point Theorem
9.	$\Box\sigma \to \Box(\Box\sigma \to \varphi)$	7, T81, IE, MD, S, Axiom (K), MP
10.	$\Box(\Box\sigma \to \varphi)$	9, 6, ____
11.	$\Box\Box\sigma \to \Box\varphi$	10, Axiom (____), MP
12.	$\Box\sigma \to \Box\Box\sigma$	Axiom (____)
13.	$\Box\varphi$	12, 6 ____, 11 ____
14.	φ	13, (____)
15.	$(\Box\sigma \to \varphi) \to \sigma$	2, ____
16.	σ	15, 5 ____
17.	$\sigma \to (\Box\sigma \to \varphi)$	2, ____
18.	σ	4, (____)
19.	$\Box\sigma \to \varphi$	17, 18 ____
20.	φ	19, 4 ____

We can sketch an elegant proof in modal provability logic of Gödel's Second Incompleteness Theorem. First, we have a modal counterpoint to the *fixed-point theorem* that yields the *Gödel sentence*:

$$\vDash G \leftrightarrow {\sim}\Box G.$$

In his letter, von Neumann noted that the consistency of Peano Arithmetic (PA) can be expressed by the formula that $(1 = 2)$ is not provable:

$$\textsc{Cons}(\mathrm{PA}) := {\sim}\Box\,(1 = 2).$$

Now the gist of the First Incompleteness Theorem is the demonstration that:

$$\text{if } \vdash G \leftrightarrow {\sim}\Box G, \text{ then } \vdash {\sim}\Box\,(1 = 2) \to {\sim}\Box G.$$

By the fixed-point theorem, ${\sim}\Box\, G$ is *logically equivalent* to G, so we have $\vdash {\sim}\Box\, 1 = 2 \to G$:

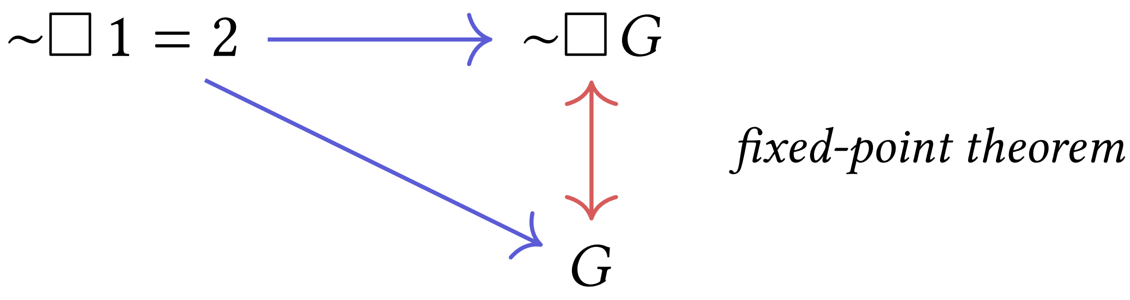


Figure 3.2

By the rule of necessitation, we may prefix a and then distribute, using (**K**), the over the conditional:

$$\vdash \Box{\sim}\Box(1 = 2) \to \Box G.$$

According to the First Incompleteness Theorem, the provability of the Gödel sentence implies the inconsistency of the system, so:

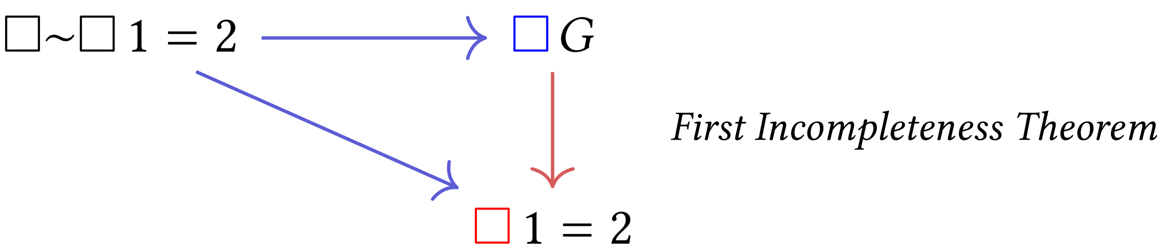


Figure 3.3

In short, we have

$$\vdash \Box{\sim}\Box(1 = 2) \to \Box(1 = 2),$$

which, by contraposition, yields

$$\vdash {\sim}\Box(1 = 2) \to {\sim}\Box{\sim}\Box(1 = 2).$$

Since $\sim\Box(1 = 2)$, by definition, is Cons(PA), we have

> Gödel's Second Incompleteness Theorem. (Cons(PA) → ~□Cons(PA)), *i.e.*, if Peano Arithmetic is *consistent*, then it cannot prove its own *consistency*.
>
> In a lecture to a joint meeting of the Mathematical Association of America and the American Mathematical Society, Gödel summarized the significance of his result for Hilbert's program: the hope of finding *"...a proof for freedom from contradiction undertaken by Hilbert and his disciples" had "vanished entirely in view of some recently discovered facts. It can be shown quite generally that there can exist no proof of the freedom of contradiction of a formal system S which could be expressed in terms of the formal system S itself ...."* (Gödel 1933b, GCW-III, p. 52).

Gödel, Kurt. 1933b. The present situation in the foundations of mathematics. *Collected Works* 3. 45–53.

3.3 Solution to Problems

Listing 3.3

Solution to Exercise Lemma

1.	~~*Show*~~ $\Box\varphi \to \Box(\varphi \to (\Box(\Box\varphi \wedge \varphi) \to (\Box\varphi \wedge \varphi)))$	3, CD
2.	$\Box\varphi$	Assume (CD)
3.	$\Box(\Box(\Box\varphi \wedge \varphi) \to \Box\varphi \wedge \varphi) \to \Box(\Box\varphi \wedge \varphi)$	Axiom (L)
4.	~~*Show*~~ $\Box(\varphi \to (\Box(\Box\varphi \wedge \varphi) \to (\Box\varphi \wedge \varphi)))$	5, ND
5.	~~*Show*~~ $\varphi \to (\Box(\Box\varphi \wedge \varphi) \to (\Box\varphi \wedge \varphi))$	7, CD
6.	φ	Assume (CD)
7.	~~*Show*~~ $\Box(\Box\varphi \wedge \varphi) \to \Box\varphi \wedge \varphi$	11, CD
8.	$\Box(\Box\varphi \wedge \varphi)$	Assume (CD)
9.	$\Box\Box\varphi \wedge \Box\varphi$	8, (**K**), MP
10.	$\Box\varphi$	9, S
11.	$\Box\varphi \wedge \varphi$	10, 6 ADJ

Listing 3.4

Solution to Exercise “Löb-Gödel Theorem”

1.	$\vdash \Box\varphi \to \varphi$	Assumption (**T**)
2.	$\vdash \sigma \leftrightarrow (\Box\sigma \to \varphi)$	Gödel’s Fixed-Point Theorem
3.	~~*Show*~~ φ	20, DD
4.	~~*Show*~~ $\Box\sigma$	16, ND
5.	~~*Show*~~ $\Box\sigma \to \varphi$	14, CD
6.	$\Box\sigma$	Assume (CD)
7.	~~*Show*~~ $\Box[\sigma \leftrightarrow (\Box\sigma \to \varphi)]$	8, ND
8.	$\sigma \leftrightarrow (\Box\sigma \to \varphi)$	2, Gödel’s Fixed-Point Theorem
9.	$\Box\sigma \to \Box(\Box\sigma \to \varphi)$	7, T81, IE, MD, S, Axiom (K), MP
10.	$\Box(\Box\sigma \to \varphi)$	9, 6, MP
11.	$\Box\Box\sigma \to \Box\varphi$	10, Axiom (**K**), MP
12.	$\Box\sigma \to \Box\Box\sigma$	Axiom (**4**)
13.	$\Box\varphi$	12, 6 MP, 11 MP
14.	φ	13, (**T**)
15.	$(\Box\sigma \to \varphi) \to \sigma$	2, BC
16.	σ	15, 5 MP
17.	$\sigma \to (\Box\sigma \to \varphi)$	2, BC
18.	σ	4, (**T**)
19.	$\Box\sigma \to \varphi$	17, 18 MP
20.	φ	19, 4 MP

4 Hao Wang's Logical Journey in Life

Abstract

Hao Wang (1921-1995) was a prolific philosopher and researcher known for his close intellectual companionship with Kurt Gödel during the last decade of Gödel's life, yet "rather little has been published on Wang's considerable body of work or on the man's personality and unusual personal history."[10] Wang spent the first 25 years of his life in China, where he studied both Chinese and Western philosophy earning an MA in philosophy at Tsinghua University in 1946 before studying at Harvard. Wang study of mathematical logic in China was sufficient advanced for him to complete his Ph.D. under Quine in less than two years. As his Chinese classmate and lifetime friend He Zhaowu remarked at Wang's memorial: "Yet his whole life from beginning to end was 'chock full of contradictions,' be it in his thought, in his studies, or in his personal life. *These contradictions were not only of his own, they belonged also to a part of the difficult course of an entire generation and people.*"[11] The goal of this article is to see what light Wang's life and philosophical work can shed on Chinese philosophy and the Western analytic tradition.

Keywords: History of Analytic Philosophy, Chinese Philosophy, Orientalism, Hao Wang, Gödel, Wittgenstein, Critical Race Theory, Asian American Philosophy[12]

4.1 Hao Wang's Journey.

On May 20, 1921 Hao Wang Hao (or, in Chinese, 王浩 *Wáng Hào*) was born into a family of intellectuals in Jinan, Shandong Province, China. Hao learned from his parents, who were teachers, both Chinese tradition and modern European-American approaches to science, art, and politics. The influence of Western approaches was brought about by Sun Yat-sen's Revolution of (1911–12) that overthrew the Qing (or Manchu) Dynasty. Wang early education was in the context the "Post-May Fourth Movement in China" (Wang's phrase in (Wang 1993), p. 40), which was inspired by the May 4th, 1919 student demonstration in Tiananmen Square to protest Japanese and Western imperialism in the aftermath of World War I.

"During the War of Resistance"[13] (the second Sino-Japanese 1937 - 1945), Hao earned B.Sc. Mathematics from National Southwestern Associated University (a consortium of Peking, Tsinghua and Nankai Universities) with the intention of using mathematics as a foundation for philosophy, which Wang pursued rather than engineering which was in vogue at the time. In 1941 Hao married Yenking, his first wife, which was the same year that both China and the U.S. entered into World War II. Wang wrote his "virgin work" in philosophy (in Chinese), a study of Hume's problem of induction in 1942, which was also the year President Franklin Roosevelt issued Executive Order 9066 authorized the force relocation and incarceration of 110,00 persons of Japanese descent living in America, 2/3 of whom were American citizens. In 1943 about 30 million people were dying of starvation in China, while in the U.S. the Magnuson Act was passed to repeal the 1882 Chinese Exclusion Act, the first immigration law in U.S. history to target a group for exclusion by race and class, given that China and the U.S. were allies.

Having taken all the courses in mathematical logic offered at his university in China, Hao agreed to form a reading group with two of his professors to work through Hilbert and Bernay's *Grundlagen der Mathematik*. After Hao presented the first chapter, however, the professors were unable to continue. Wang notes, "*I also became lazy... [and] it wasn't until seven or eight years later, when I was going to teach*

[10] Gary R. Mar is an Associate Professor of Philosophy at Stony Brook University where he is also the Founding Directors of the Philosophy Department Logic Lab and the Asian American Center. He was the last dissertation student of the 20th century logician Alonzo Church and co-author with Kalish and Montague of the logic textbook *Logic: Techniques of Formal Reasoning* (second edition, OUP) and the catalyst for the expansion of the APA Committee on Asian and Asian-American Philosophers and Philosophies. Email: Gary.Mar@stonybrook.edu. Charles Parsons and Montgomery Link in their preface to *Hao Wang: Logician and Philosopher*, a collected work published 15 years after Wang's death, p. 1.

[11] He [1995], translated in Parsons and Link [2011], 49, italics mine.

[12] This paper is the result of participating in a panel "The Analytic Tradition and Chinese Philosophy" at the 2016 Eastern Division Meeting of the APA sponsored by the Committee for Asian and Asian-American Philosophers and Philosophers. For the history of the usually long name of this committee, see (*APA Newsletter on Asian and Asian-American Philosophers and Philosophies*, Spring 2003 (vol. 2, no. 2), 2).

Wang, Hao. 1993. From kunming to new york.

[13] Wang's phrase in Wang [1993].

a class on that subject, that I finished reading the whole book."[14] Wang's Chinese writings include "The metaphysical system of the New Lixue" [1944], "Language and Metaphysics" [1945]. Wang characterizes the time of his education as during the "Post-May Fourth Movement" which was characterized with disillusionment with the Chinese Republic and traditional Chinese culture to protect China from imperialism. A recurrent theme of Wang's Chinese writings was the dilemma facing Chinese intellectuals in light of the Opium Wars (1839 –1842, 1856 – 1860).

In 1945 Wang earned his M.A. in Philosophy at Tsinghua University. "During the defense," Wang recalled, "Professor Shen Youding asked me why I wanted to study philosophy. I said I was interested in the human dilemma. Professor Shen said, 'In the West it is literature that focuses on the human dilemma, not philosophy." This was the same year that Truman ordered atomic bombs to be dropped on Hiroshima (Aug. 6th) and Nagasaki (Aug. 9th) after which Japan surrendered (Aug. 15th).

The following year at the age of 24 Wang left China on a U. S. State Department scholarship, to study philosophy at Harvard.[15] Given his background in mathematical logic in China, Wang was able to complete his dissertation *An Economical Ontology for Classical Arithmetic* under Quine in one year and eight months, becoming Quine's fifth dissertation student. Wang then became a Junior Fellow of the Society of Fellows at Harvard. During this time Wang was able to find an elegant repair for Quine's inconsistent first edition of *Mathematical Logic* [1946] and demonstrate the consistency of his proposal relative to Quine's *New Foundations,* a repair adopted and acknowledged by Quine in the second edition (Quine 1951). In October of 1949 Mao Zedong declared the existence of the People's Republic of China (PRC), and China, who had been an ally, became a communist threat. McCarthyism created a climate of anti-Communism domestically and the U.S entered into the Korean War (1950-53).

During the academic year 1950-1951, Wang studied in Zurich under the auspices of Paul Bernays, where he pursued the philosophy of mathematics, with a specialization in the type of predicative mathematics advocated by Bernays. Wang worked on questions of the relative strengths of axiomatic set theories and pioneered the revival of Hermann Weyl's work on predicative mathematics, which takes its motivation from Russell's "Vicious-Circle Principle." In 1953 Wang contributed his own principle of *autonomous iteration* as a natural extension of *predicativism.*

In the Spring of 1955 Wang was invited to be the second philosopher to deliver the prestigious John Locke Lectures at Oxford and then became a reader in the Philosophy of Mathematics at Oxford. This period coincided with the beginning of U. S. involvement in Vietnam (1955 – 1975). In 1956 Wang received a letter from Chancellor Ma Yinchu of Peking University offering him a teaching position, but Wang declined at the time because of interest in pursuing philosophy. Two years later the Cultural Revolution forced Chancellor Ma to resign making it impossible for Wang to retrun to China. Jane Wang (*Hsiaoching*) was born in Oxford to Hao and Yenking in 1957.[16]

In the Spring of 1958 Wang arranged for his teacher Professor Jin Yuelin who had promoted logic in China, to report to a meeting of Oxford philosophy professors in which Jin explained why he had abandoned academic philosophy and become a Marxist. Wang, who wrote beautifully but whose English could be difficult to understand, remarked, "*Most of the professors who heard the lecture felt his proof a bit too simplistic. But because Professor Jin's British English was especially elegant and polished, the majority of Oxford professors treated him with utmost respect."*[17] At this time, the Soviet Union, under the influence of Marxist philosophy, denounced mathematical logic *"as a conceptual game of capitalist class idealism".*[18]

Upon returning to Harvard, Wang offered a professorship, not in Philosophy where most of his contributions, research and passion had been devoted, but in Applied Mathematics. Ahead of his time, Wang found ways to create academic partnerships with the emerging computer industry. While at IBM Wang (Wang 1961) invented a representation of formal systems known as Wang Tiles, graphically represented by square tiles with a color on each side. Wang's student Robert Berger (Berger 1966) demonstrated in his dissertation the Undecidability of Wang's Tiling Problem by showing its decidability would contradict Turing's proof of the *Unsolvability of the Halting Problem.* Berger also found a counterexample to Wang's conjecture that there were no aperiodic Wang Tilings.[19] Wang popularized his geometrization of logic in his *Scientific American* article, "Games, logic and computers" (Wang 1965).

[14] Wang [1982], Parsons and Link [2011], 31.

[15] The Chinese Exclusion Act [1882], and its various amendments (strengthened in 1884 and expanded by the Scott Act (1888)) and renewed (the Geary Act (1982) and again in 1902 without termination date) made Chinese immigrants permanent aliens by excluding them from U.S. citizenship, so that Chinese men in America had little chance of starting families in America. Although the Chinese Exclusion Act was "repealed" by the Magnuson Act (1943), this act only raised the quota of 100 Chinese per year from anywhere in the world to 105 as compared to a quota of 65,721 immigrants from Great Britain and Northern Ireland. In preparation for entering into World War II, through the "Lend-Lease" program, President Franklin D. Roosevelt had $1.6 billion to invest in China.

Quine, Willard van Orman. 1951. Two dogmas of empiricism". *Originally in**The Philosophical Review* 60. 20–43.

[16] Jane would grow up and become a jazz musician (Boston) and her brothers San-You and Yi-Ming became a doctor (Boston) and an astrophysicist (Washington), respectively.

[17] Wang [1982], in Parsons and Link [2011], 35.

[18] He [1995] in Parsons and Link [2011], 50.

Wang, Hao. 1961. Proving theorems by patternrecognition—II". **Bell System Technical Journal** 40**1. 1–41.

Berger, Robert. 1966. The undecidability of the domino problem,"*memoirs. *The American Mathematical Society** 66. 72.

[19] Computational ideas have been a catalyst for contemporary philosophical investigations and speculations, e.g., Wang Tiles serve as the plot engine for a short story "Wang's Carpets" by Australian science fiction writer Greg Bear [1995], a reflection on the nature of reproduction, life, and intelligence in a post-human context.

Wang, Hao. 1965. Games, logic and computers. **Scientific American** 5(v.). 98–106.

Becoming an American citizen in 1967,[20] Wang moved his academic home from Harvard to Rockefeller University where he headed a research group in logic and computation. Wang began corresponding with Gödel while helping select and clarify various passages on the logical writings of Skolem for Jean van Heijenoort's classic anthology *From Frege to Gödel*.[21] In October of that year Wang began his weekly conversations with Gödel at the Institute for Advanced Studies at Princeton. The first series of conversations from 1971 - 1972 influenced significant parts of Wang's *From Mathematics to Philosophy* (Wang 1974), which were published subject to Gödel's approval. The second series of conversations with Gödel began in the Fall of 1975. These remarkable conversations are chronicled and commented upon in four of Wang's books beginning with *From Mathematics to Philosophy* (1974) and continuing in *Beyond Analytic Philosophy* (Wang 1986), *Reflections on Kurt Gödel* (Wang 1987) and his posthumously published *A Logical Journey: From Gödel to Philosophy* [1997].

[20] This was two years after the Hart-Celler Naturalization and Immigration Act (19654), which removed the quota system and, for the first time in U. S. history, put Asian immigrants on an equal footing with other nations.

[21] van Heijenoort [1967], vii.

Wang, Hao. 1974. *From mathematics to philosophy*(humanities press. New York.

Wang, Hao. 1986. *Beyond analytic philosophy* Cambridge, Mass.: MIT Press.

Wang, Hao. 1987. *Reflections on kurt gödel*. Cambridge, Mass.: MIT Press.

Even with the publication of Solomon Feferman's *et al.* monumental 20-year project, the *Collected Works of Kurt Gödel*, vols. I–V [1986 - 2003], Wang's pioneering books remain significant because they provide the philosophical background missing in Gödel's published papers and even in his *Nachlass*. The philosophical themes or threads that run through Gödel's work remain unstated not because they were secret but because they were so interwoven into the fabric of his thought that it was unnecessary for him to spell them out. Wang books on Gödel made *mathematicians* aware of the *philosophical* views that led to Gödel's remarkable theorems and showed *philosophers* how philosophical views could be made *mathematically* rigorous. Wang's accounts of his conversations with Gödel [1974, 1985, 1987, 1996] remain invaluable to philosophy. In China Hao Wang "was publically acknowledged as the inheritor of the mantel of Gödel", but as Parsons and Link add in a footnote "an acknowledgement that would not extend globally."[22]

[22] He [1995] in Parsons and Link [2011], 50, footnote 4.

4.2 Hao Wang's Contradictions in Life.

Charles Parsons and Montgomery Link, in their editorial preface to *Hao Wang: Logician and Philosopher* [2011], a memorial volume not assembled until 15 years after Wang death, make an intriguing observation:

> Clearly [Wang] never lost his identification as Chinese, and we think he resisted being categorized as Chinese-American or Asian-American, although he did become an American citizen in 1967. The influence on him of Chinese culture and thought comes through in a number of his writing, maybe especially in his reaction to the very Western rationalism of Kurt Gödel, which fascinated him but which he could never fully embrace.

There is no reason to question the editors' opinion with regard to Wang's preferences about terms he would use to express what we would today call his "identify politics." The asymmetry of the hyphenated terms "Chinese-American" or "Asian-American" typically implies that one is referring exclusively to a type of American. However, a different question can be raised. We can use terms with solidus (e.g., "Chinese/American" or "Asian/American") to refer to Wang's status as an individual of Chinese or Asian descent residing in America. What facts about the racial formation of Asian/American intellectuals during Wang's lifetime shaped their life prospects and career possibilities?

Charles Parsons has, perhaps, done more than any other philosopher to bring the unduly neglected accomplishments of Hao Wang to the attention of the APA. However, Parsons' repeated remarks on of Wang's "*Chineseness*" may hide the complexities of racial formation faced by Wang and his generation. Parsons notes that Wang essentially thought of himself "as simply Chinese, a member of the *Chinese diaspora* that has existed for centuries."[23] Parsons characterized Wang's extensive exchanges with Gödel as "conversations between two thinkers with synoptic ambitions, one of whom sought resources from earlier Western traditions, and the other of whom, however Western his training and career, *never ceased to be Chinese*."[24] Parsons characterized Wang's responses to Gödel's discussions of *Weltanschauung* (e.g., of pre-Kantian rationalism and Leibnizian optimism) as "show[ing] his *Chinese cultural background*."[25]

[23] Parsons [2002] 28, *italics mine.*

[24] Parsons, op. cit., *italics mine.*

[25] Parsons [2011], 78, footnote 30, *italics mine.*

The characterization of Chinese in America as part of the global diaspora of *"Overseas Chinese"* is often used to explain why Chinese failed to assimilate and to become American. This *cultural* explanation blames culture for the isolation and marginalization that Chinese/American[26] intellectuals faced and ignores the more subtle forces of racial formation operating on Oriental scholars in American academic institutions. The construction of the Oriental as a "*sojourner*" in the West whose life goal of the triumphant journey home often functioned as a rationalization. This dream of returning home served as a powerful defensive mechanism for many Chinese/American intellectuals coping with social isolation and professional discrimination.

[26] The solidus notation in 'Chinese/American' is here used *inclusively* for being both Chinese and American and avoids the asymmetry of the label 'Chinese-American' which typically refers *exclusively* to a type of American.

Professional Orientalism in the *careers* and *outlook* of academics of Asian descent in America has been examined by Henry Yu in *Thinking Orientals* (Yu 2001) a sociological study of historical transition from "gentlemanly to institutionalized Orientalism" experienced by intellectuals of Chinese and Japanese descent living in the American from 1920 – 1965. Yu's study focuses on the Chicago School of Sociology, spear-headed by Robert Ezra Park (1864–1944), who with the help of his mid-westerner colleagues, recruited Oriental scholars, but the same sociological forces and racial formations were faced by other academics of Asian descent.

Yu, Henry. 2001. _*Thinking orientals: Migration, contact, andExoticism in modern america*_. New York, NY: Oxford University Press.

In Chapter "Strangers from the Midwest: Robert Ezra Park and Other Men with Three Names" of *Thinking Orientals*, Yu explains why these Midwestern sociologists were so interested in Orientals and how their outlook was conditioned by orientalism. The conceptual link between the European *white immigrant* and the Negro *non-white non-immigrant* was provided by the Oriental, who was both *immigrant* and *non-white.* The "Oriental Problem" was to explain the cultural inability of Orientals to overcome the last stage of the *assimilation* cycle, which the Chicago School equated with becoming *American.* Park and his associates sought an explanation of the failure of Orientals to become American through a cycle of assimilation, not in terms of *racial formation,* but in terms of *cultural distance.*

Robert Park, blinded by his culturalist assumptions, came to think of the Oriental face as a mask that could disguise the inner thoughts and feelings, which is strikingly evidence in the following passage: *"I recently had the curious experience of talking with a young Japanese woman who as not only born in the United States, but was brought up in an American family, in an American college town, where she had almost no association with members of her own race.... I found myself watching her expectantly for some slight accent, some gesture or intonation that would betray her racial origin... When I was not able, by the slightest expression, to detect the Oriental mentality behind the Oriental mask, I was still not able to escape the impression that I was listening to an American woman in a Japanese disguise."*[27]

[27] Yu [2001], 67.

Quine in his autobiography *The Time of My Life* characterizes Wang as "persistently unhappy" during his time at Harvard. This begs the question: did Hao Wang have reason to be unhappy? Recall that Wang, having already studied mathematical logic in China, wrote his dissertation under Quine in under two years. Wang had repaired Quine's inconsistent system in *Mathematical Logic*, had spent five years at Oxford where he was invited to be the second philosopher to give the John Locke Lectures. The topic of Wang's lectures was "On Formalizing Mathematical Concepts." Wang had been preceded by O. K. Bouwsma and was followed by a distinguished list of philosophers: A. N. Prior, A. C. Jackson, Gregory Vlastos, Nelson Goodman, Jaakko Hintikka, Wilfred Sellars, Paul Lorenzen, Noam Chomsky, Donald Davidson, Saul Kripke, Hilary Putnam, and so forth. Wang's time at Oxford overlaps with his first period of engagement with the philosophy of Wittgenstein (1953 – 1958).

Upon returning to Harvard, Wang as offered a position, not in Philosophy, but in Applied Mathematics. Juliet Floyd reports: *"In conversation Wang told me that one reason he had found his professorship at Harvard unsatisfying is that he was appointed in the Department of Applied Mathematics, and not in Philosophy, where he felt the bulk of his efforts and his contributions had been made."*[28] Quine was a mid-westerner with roots in Akron, Ohio with three names, a fact that he proudly details in *The Time of My Life* (1986): he was the child of Colye Robert *Quine* and Harriet ("Hattie") Ellis *née van Orman* named after a mathematician.[29] One wonders why W. V. O. Quine did not intervene on Wang's behalf.

[28] Floyd [2011], 157 footnote 17.

[29] See http://www.quine.org/crq-tree.html.

Juliet Floyd notes: "Wang's role in shaping more than one generation's understanding of the fundamental problems in logic and their history through the 1950s was significant, and not as widely acknowledged as it should be.... Because Wang taught at Harvard and Oxford, the cumulative impact of his teaching on the dissemination of logic was significant.... Hide Ishiguro has stressed to me how supportive Wang was of Michael Dummett during his early years teaching at Oxford, when logic was not a very popular subject among philosophers there."[30] Among Wang's students who went on to have distinguished careers as philosophers and logicians were Charles Parsons, Burton Dreben, Michael Dummett, Donald A. (Tony) Martin, and Hilary Putnam.[31]

[30] Floyd [2011], 146.

[31] Floyd ([2011], 87) herself regarded Wang as a mentor who provided her with an authentic example of what the Chinese call *xiansheng* (先生, teacher, master).

Wang expressed his dissatisfaction with analytic philosophy in *Beyond Analytic Philosophy: Doing Justice to What We Know* (Wang 1986). Quine, who described himself as a "disciple of Carnap for six years," declared his independence from Carnap his celebrated "Two Dogmas of Empiricism" (Quine 1951) [1951, 1961].

Wang, Hao. 1986. *Beyond analytic philosophy*. Cambridge,Mass: MIT Press.

Quine, Willard van Orman. 1951. Two dogmas of empiricism. *Originally in*The Philosophical Review** 60. 20–43,.

[DATE]. Similarly, Wang playfully labeled Quine's philosophy "Logical *Negativism*" (a contrast that plays off of Carnap's "Logical *Positivism*") which captures Quine's peculiar combination of *local precision* and *global indefiniteness*, supported by such *negative* Quinean dogmas as the "*indeterminacy* of translation," the "*inscrutability* of reference," and, to borrow Gilbert Harman's phrase, the "*death* of meaning." Supported by vaguely enunciated pragmatic appeals to semantic holism, Quine's logical negativism not only "failed to do justice to what we know"—but, ironically Wang noted, failed to give an adequate account of logic, mathematics, and science, the disciplines from which Quine's philosophical positions drew their rhetorical strength. Wang's criticism of Quine was frustrating to analytic philosophers trained to publish "piecemeal exercises" of focused argument analysis and to diagnose arguments in terms of precisely defined logical failings.

Instead of adopting a *piecemeal* approach, Wang's writings aimed at *perspicuity*—clarity of the sort that can be found in an elegantly arranged mathematical proof or an insightfully arranged set of quotations. Wang was known for his ability to locate a philosophical position in *intellectual history*—a history not told dramatically but with discerning and judicious insight.[32] Floyd characterizes Wang's *synoptic* style of argument: "When arguments are found in Wang, they are deductions from his observations, or relative comparisons of the interest and correctness of different responses, concepts, or principles. The tone of the whole is always tentative, pluralistic, and ... synoptic rather than systematic in aim."[33]

[32] Floyd [2011], 156.

[33] Floyd [2011], 18.

Wang was concerned that philosophy "do justice to what we know." He characterized his major statement on the philosophy of mathematics, *From Mathematics to Philosophy*, not in terms of *constructing* a grand *theory*, but in terms of seriously *collecting data*:

> This book certainly makes no claim to be philosophical theory or a system of philosophy. In fact, for those who are convinced that philosophy should yield a theory, they may find here merely data for philosophy. However, I believe, in spite of my reservations about the possibility of philosophy as a rigorous science, that philosophy can be relevant, serious, and stable. Philosophy should try to achieve some reasonable overview. There is more philosophical value in placing things in their right perspective than in solving specific problems. Both hasty speculations and piecemeal persistence on artificial issues tend to hamper cumulative progress in philosophy.[34]

[34] Wang [1974], x.

As an active researcher in the emerging field of computational logic, Wang was able to see the impediments imposed by Quine's overly narrow analytic conception of logic pronouncing limitations on the proper scope of philosophical and logical inquiry. Wang, in his contribution to Quine's "Schilpp volume" one year after his critique in *Beyond Analytic Philosophy*, sharpened his critical assessment: "for the working logicians, much of Quine's work is thought to be off the mainstream."[35]

[35] Wang's contribution in Schilpp [1986], 635.

Eckehart Köhler was scholar interested in Gödel's Platonism, who first contacted Wang in 1979 on the advise of Solomon Feferman, and who later in 1986 as founder of the Kurt Gödel Society in Vienna invited Wang to become its first president. Köhler in his "Collaborating with Hao Wang on Gödel's Philosophy" (Köhler 2011) in the memorial volume recounts a revealing with Wang:

Köhler, Eckehart. 2011. Collaborating with hao wang on gödel'sPhilosophy. (Parsons & Link, Eds.).

> I began explaining to Wang the well-known work of Herbert Simon, a major figure in Artificial Intelligence. He and his students had done some of the best work on scientific discovery, using AI methods. (I had at other times tried explaining related concepts of decision theory, statistical weighing of evidence, concepts of probability, in attempting

> to justify Gödel's famous claim that mathematical knowledge is analogous to empirical knowledge, especially.... Wang practically exploded. He was disgusted with Simon and did not want to hear a thing about him, and considered him an execrable logician. Wang would brook not contradiction, so I changed to topic (Köhler adds in a curious footnote: "Oddly enough, Simon... spent quite a lot of time in China and had learned Chinese.")

Köhler speculates on the source of Wang's anger:

> Without knowing any details of Wang's work in computer proofs, I am sure that he was a logician vastly superior to Simon, and I suppose he got sick of hearing form AI researchers that Simon was allegedly the first to do an AI logic proof. I am also sure that Simon was would agree with Wang that the proof he presented at the famous 1956 AI-conference was utterly trivial and without much inherent merit.

Let's examine some of the logical and historical details before drawing any conclusion about Wang's anger. Newell and Simon wanted to program a computer to carry out heuristic problem solving through symbol manipulation, and they considered such problems as playing chess, the solving of geometrical problems, and, as an afterthought, proving logical theorems. In 1955 Simon claimed "over Christmas Allen Newell and I invented a thinking machine"[36] because he had simulated "by hand" a proof from Whitehead and Russell's *Principia Mathematica*, but it wasn't until August 1956, that the *Logic Theorist* program actually run on RAND's Johnniac computer (named after von Neumann) completed the proof of theorem 2.01.

[36] McCorduck [1979], 116.

The *Logic Theorist* was given a list of axioms and definitions and kept in memory a list of previously proved theorems. When it is given a new logical theorem to prove, it runs through all the operations of which it is capable searching for a proof. Consider a proof of theorem 2.01:

$$2.01 \vdash \neg(p \rightarrow \neg p) \rightarrow \neg p$$

The program would have had as an axiom

$$\vdash (p \vee p) \rightarrow p,$$

which by a rule of substitution of ~p for p could obtain

$$\vdash (\neg p \vee \neg p) \rightarrow \neg p,$$

from which, in turn, by a rule of substitution of equivalents, in particular, an instance of the definition

$$p \rightarrow q \ :=: \ \neg p \vee q$$

could obtain the required theorem.

It should be kept in mind that Newell and Simon *structured* the problems given to the program, including the order in which they were given. In other words, it could be argued that the *Logic Theorist* did not by any stretch of the imagination show that one could mechanize Gödel's notion of intuition: the program was only doing what it had been programmed to do in a very limited realm of problem solving. Eventually the *Logic Theorist* was able to prove 38 of the first 52 propositional theorems. About half the proofs accomplished in less than a minute, most of the remainder taking between 1 – 5 minutes, some taking as much as 15 – 45 minutes.

How did this compare with Wang's accomplishments? In 1959 Wang programed an IBM 704 computer to prove all the logical theorems (over 350, including theorems involving not only propositional logic but also predicate logic and identity) of *Principia Mathematica* in less than 9 minutes. In contrast to Herbert, who triumphantly announced his accomplishment as showing the power of AI to simulate human logic, Wang regarded his accomplishment as demonstrating the "essential lack of conceptual richness" of Russell and Whitehead's *Principia Mathematica*.[37] Wang help to define the new research area of ATP (automated theorem proving) and to show how it rested upon results in mathematical logic.

[37] Floyd [2011], 164.

This work would be the basis of his later work (Wang [1960, 1961, 1963]) on the ∀∃∀ case of Hilbert's *Entscheidungsproblem*. Instead of stocking the computer program with *ad hoc* "heuristics," a research program which has since gone out of style, Wang programmed the "cut-free" formalisms of Herbrand the Gentzen and emphasized the importance of algorithmic pruning, which could eliminate useless terms in advance. Furthermore, Wang provided a well-conceived list of theorems of predicate calculus that could serves as "benchmarks" for judging the effectiveness of new theorem-proving programs. For these achievements Wang would eventually be awarded in 1983 on the recommendation of a committee composed of David Luckham, John McCarthy and chaired by Martin Davis the first *Milestone Prize for Automated Theorem Proving* (ATP) by the *International Joint Conference on Artificial Intelligence*.

To return to Köhler account, Köhler found Wang's reaction to be inconsistent:

> A few days after the clash on Simon, while musing on Gödel's notion of intuition, I mentioned how close it seemed to what AI people (following Simon) now call heuristics; … Wang again rebuffed me, calling this too "scientistic." I slyly hinted that his refusal to consider the idea violated his own injunction "to do justice to what we know." Wang of course immediately got the point and did not reply….

Perhaps even more inconsistent is the suggestion that Wang's silence indicated his *capitulation* to Köhler's criticism. One cannot dismiss the *legitimacy* of Wang's anger on the grounds of Wang's *incivility* or Wang's *silence*. Anger towards institutionalized discrimination can be a legitimate and healthy response. It is presumptuous for more privileged parties to assume that unrecognized parties expressing anger "be civil" before their complaints can be given a hearing, especially when this requires, in effect, that they case cease making forceful points about unfairness. Silencing of those in subordinate positions protects institutionalized discrimination by removing any record of wrong. The vicious circle principle of silencing can work in two ways—when one's colleagues dismiss one as a knower with a legitimate perspective due to one's anger and when one in frustration perceives one's colleagues as unwilling or unable to listen to one's testimony and so engages in self-censoring.

This *vicious silencing principle* is a form of racial formation is often at play in academic circles. This principle combines stereotyping with an asymmetrical application of a principle of civility. When someone from the dominant group is uncivil, this very incivility is valorized as being "forthright" or "refreshingly honest" or even "courageous." However, when those from a subordinated group are uncivil, perhaps uncharacteristically, this angry outburst is used as grounds for not taking their complaints seriously. For example, the silencing response "I can't hear you when you're angry" to Black or women philosopher is now widely acknowledged as ways of illegitimately delegitimizing the criticism.

Oientalism functions not only professionally but personally. Scholars of Asian descent establishing themselves as academics in America during Wang's generation would often *re-orient* their lives selectively around acceptable Asian values (e.g., Chinese culture and philosophy) while distancing themselves from cultural traits considered to be un-American (e.g., clannishness and indirectness). Ever since his youth Wang conceived of philosophy as a more holistic enterprise: "One might wish to say that the task of every philosopher is to describe eventually 'the world as I see it' and that his achievement is to be evaluated by how significant a picture he gives."[38]

[38] Wang [1974], 334.

Henry Yu's book takes as a case study Rose Hum Lee (1904–1964), a first generation Chinese American, who was the first *woman* to become chair*man* in an American university (to succeed Lee was forced not only to become more *modern* but also more *masculine*). In research papers, Lee adopted the cultural analysis of alienation of the Chicago School of sociology and wrote objectively:

> *To retaliate for not heeding their wishes, they spread tales of her supposed misdeeds in the Midwest, where she attended a famous university. The Chinatowners of this prairie city were delighted to have a plum to pick. No native-born of this group had ever obtained a doctorate*

in philosopher, though some received their master's and medical degrees… The local Chinese, instead of being sympathetic to professional attainments, disparaged this girl's achievements. Her personal life was the subject of slander, gossip, envy and conspiracy. There were no congratulations when her doctorate was awarded.

Perhaps Wang's self-representation of himself as a "Chinese in exile" was conditioned by the discrimination he faced professionally and personally, a self-presentation that evolved during the two thirds of his life that he lived in America. While Wang was still entertaining thoughts of returning to China—after the birth of the PRC in 1949 but before the Cultural Revolution in China (1966 – 1976) brought about the public persecution of intellectuals—he shifted his research focus to computation with the thought that programming skills would be more useful to China than his research in philosophy. Martin Davis ([2011], 76) in the midst of his account of Wang's contributions to Automatic Theorem Proving and to Hilbert's *Entscheidungsproblem*, recalls a turning point in Wang's disillusionment:

> Like to many others, he eventually became disillusioned with the Maoist order. I remember particularly his telling me of his astonishment at a letter severely criticizing him that he received from his father, a secondary school teacher in China, during the Cultural Revolution. He understood with considerable distress that his father would have written such a letter only under great pressure.

Wang reminisced about how he and his classmate in China He Zhaowu visited each other in 1980 in Kunming and in New York:

> *I began to doubt my own most recent views. I had imagined certain facts that were grounded in wishful thinking; upon this foundation I had erected a logical construction that really was a castle in the sky….. I remember the pain that bitterness I suffered at the time because of the wrenching transformations of my thinking and the lost of my convictions."*[39]

[39] Wang [1933] in Parsons and Link [2011], 41.

Influenced by Zhaowu's presentation of the poetic pentology of *Bodhisattva Barbarians*, Wang wrote a personal essay about *"homesickness in a foreign land"* but did not publish it regarding it, in the end, as *"sick with self-pity"* and *"nauseating."*[40]

[40] Wang [1992] in Parsons and Link [2011], 44.

4.3 Hao Wang's Philosophical Journey.

Wang believed that "Philosophy has a longer history and a larger diversity of traditions to select from" (Wang [1986], 192). As an *analytically trained* philosopher who was an active researcher in logic, Wang contributed an important *diagnosis* of the *misconceived* role of logic in Anglo-America analytic philosophy. Wang was in a position to comment knowledgeably on the contradictory roles *logic* played in Quine's analytic philosophy: logic was at once *excessively emphasized* and *ineffectively employed*:

> Directly and indirectly logic plays an important part in much of contemporary Anglo-American academic philosophy…. The way logic is commonly used in philosophy seems to me to do less than justice to the full richness of logic as a study of the foundations of mathematics; and that excessive emphasis on the importance of logic for philosophy, combined usually with a misapplication of logic, seems to me to have led to a far from balanced view of philosophy, especially as it is understood in the traditional sense. Moreover, the much-publicized juxtaposition of logic with positivism (or empiricism or 'analytic' philosophy) has burdened logic with a guilt-by-association, resulting in a surprising ignorance of logic on the part of philosophers of other persuasions.[41]

[41] Wang [1976], ix.

As a *Chinese philosopher*, Wang was attached to a broader view of the philosophical enterprise "beyond analytic philosophy." In a concluding chapter entitled "Metaphilosophical observations" of *Beyond Analytic Philosophy*, Wang acknowledges the Chinese influences on his conception of philosophy: "… my professional training is nearly all in Western philosophy (much of it even logic-oriented), yet my formative years were lived in China. I have tried hard but have not been able to shake off my early conviction that philosophy is not just one subject more or less like any other, but something special. Even today such a belief, I think persists in China.... I continue to believe that philosophy should somehow be comprehensive and aim at a unified ... outlook.... I find myself attached to the Chinese tradition of mixing together philosophy, literature, and history—a tradition that conditions and is conditioned by the central concerns of its philosophy; the interest in politics ties it and the concern with the unity of nature and person overlaps with art and literature."[42]

[42] Wang [1986], 194.

To advance the dialogue on such topics as "The Analytic Tradition and Chinese Philosophy," we need to overcome the *duality* of viewing philosophy in terms of "East versus West"—in particular, the unexamined tendency to view the relationship in terms of "*challenges*" or "*contributions*" to the mainstream. "Challenges" presuppose a false philosophical polarity, e.g., in discussing Eastern versus Western ethics in terms of polar concepts of *shame* versus *guilt* tends to stereotype the meaning of each and to discourage remembering cases of ethical reasoning or traditional virtues in which both qualities are present. "Contributions" presuppose one tradition as the philosophical *mainstream* thereby marginalizing the other as making "contributions."

Orientalism shaped the *disciplinary* self-understanding of philosophy. A kind of *professional orientalism*, also shaped the careers of many academics of Asian descent living in America (as well as England and Europe) as well as their *autobiographical* self-understanding of themselves as academics. Edward Said famously coined the term *orientalism*,[43] which is used by scholars of Cultural Studies for a type of depiction of the Orient (East Asia, South Asia, and especially, for Said, the Middle East) by writers, artists and scholars from the West. While Said's use of the term was far too sweeping and impressionistic, scholars in Asian American Studies have more carefully articulated various forms of orientalism.

[43] Edward Said's *Orientalism* (Vintage Books, New York, 1978), was *catalytic*, if not always philosophically *accurate*. Said's philosophical misconceptions were elaborated in my presentation "Critique of Orientalism" delivered at the Annual Meeting of the Association for Asian American Studies, April 24, 2009.

An essential aspect of Said's thesis was that orientalism was essential to the West's self-understanding and self-promotion. Consider Hegel's claims in his *Lectures on the History of Philosophy* (1825–1826):

> [In the East] conscience does not exist, nor does individual morality. Everything is simply in a state of nature, which allows the noblest to exist as it does the worst. The conclusion to be derived from this is that no philosophic knowledge can be found here.... The Eastern form must therefore be excluded from the history of philosophy.... Philosophy proper commences in the West.[44]

[44] Hegel [1882] in Haldane and Simon translation [1974], 97-98.

Even Bertrand Russell's more liberal lament over Great Britain's colonialist policies in *The Problem of China* (1922) appeals to orientalism to find cultural reasons for the dominance of the West:

> The British view is still that China needs a central government strong enough to suppress internal anarchy, but weak enough to be always obligated to foreign pressure.... Possession, which is one of the three things that Lao-Tze wishes us to forego, is certainly dear to the heart of the average Chinaman. As a race, they are tenacious of money—not perhaps more so than the French, but certainly more than the English or the Americans. Their politics are corrupt, and their powerful men make money in disgraceful ways.... Nevertheless, as regards the other two evils, self-assertion and domination, I notice a definite superiority to ourselves in Chinese practice. There is much less desire than among the white races to tyrannize over other people. The weakness of China internationally is quite as much due to this virtue as to the vices of corruption ....[45]

[45] Russell [1967], 552.

Wang himself noted the historical, if not conceptually necessary, collusion between Great Britain's imperialism and the aesthetic and autonomous morality embraced by its intellectuals:

> In the heyday of British imperialism, a rather influential ethical doctrine came out of Cambridge which signaled out esthetic enjoyments and personal affection as good in themselves. In theory, this doctrine does not reject the suggestion that for a long time to come eliminating miseries will be of more moral value than pursuing personal enjoyments. In practice, it has had the effect of enhancing the complacency of intellectual aristocrats.[46]

[46] Wang [1974], 329–30.

In 1985 Wang received honorary doctorates from Peking (China's 'Harvard') and Tsinghau (China's 'MIT') Universities, and was again seriously considering returning to China. Although China was reversing the repressions of the Cultural Revolution under Deng Xiaoping, Wang "sorrowfully torn up his invitation" to teach and do research in Beijing after the 1989 Tiananmen Square massacre.[47]

[47] Köhler [2011], 58.

Racial intermarriage was of special interest to the Chicago School of Sociology because in their minds it was ultimate proof of assimilation and hence, in their view, of becoming American. However, it should be kept in mind that anti-miscegenation laws were in effect since before the United States was a republic and remained in effect until ruled unconstitutional by the U.S. Supreme Court in 1967. For example, prior to the Cable Act of 1922, women, but not men, lost their U.S. citizenship if they married a foreign spouse. The Cable Act guaranteed such women could retain their citizenship but only if married to an "alien eligible to naturalization." Since at the time of the act's passage, Asian aliens were not considered to be racially eligible for U.S. citizenship, any woman who married an Asian alien lost her U.S. citizenship.

Around 1988 Wang met Hanne Tierney[48] at an invitation-only gathering of artists, academics and other accomplished persons called the "Reality Club." Hanne Tierney, who would become Hao's third wife, was a German immigrant, who lived in Prague and journeyed to America by way of England at the age of nineteen. She authored a successful children's book about her experience in America and then created an *avante-garde* style of puppetry that offered her a three-dimensional language of sculpture. According to Hanne, Hao characterized the difference between herself and himself: "You are an *immigrant*, I am an *exile*." Hao once explained to her that "China was God", that is, China served much the same purpose for exiled Chinese as God did in Western religion: what overseas Chinese desired from China was *forgiveness*, and the *good deeds* they performed in a hostile environment were done for China. The "Death of God" explains the depth of suffering caused by Wang's loss of his faith in Marxism and Maoism in 1979. Hao and Hanne visited China together in 1992.

[48] After reading Wang's obituary in the *New York Times*, I contacted his surviving wife Hanne Tierney, who provided these recollections.

4.4 Wang, Wittgenstein, and Gödel.

Wang approvingly quotes a mimeographed manuscript of Yueh-Lin Chin [1981]: "Chinese philosophers were all of them different grades of Socrateses."[49]

[49] Wang [1986], 194.

Towards the end of his life, Wang continued to work on a manuscript for *Gödel, Wittgenstein and Purity of Mind: Logic as the Heart of Philosophy* but had great difficulty in completing his manuscript and set the project aside. Despite his reservations about Gödel's overly optimistic faith in the "possibility of philosophy as a rigorous science," Wang unreservedly praised Gödel's way of life:

> *Gödel exemplified, I think, a way of life and work that inspires a greater faith in reason, questions the 'prejudices of the time' and stirs our imagination to strive for more autonomy by examining our largely derivative sense of what is important in life.*[50]

[50] Wang [1987], ix.

While Wang never became a *Wittgensteinian*, he seriously engaged Wittgenstein's *philosophical* insights. Wang's second period of engagement with Wittgenstein's philosophy spanned the years 1981 – 1995. Wang admired Wittgenstein's life-long struggle to achieve *perspicuity*: "*One main difference between Wittgenstein and most contemporary Anglo-American academic philosophers would seem to be the indulgence of the latter in clever, small arguments clouded by all sorts of extraneous detail.*" [51]

[51] Wang [1976], 348.

Let's summarize some of our conclusions thus far. As a *Chinese/American* philosopher, Wang was able to pursue philosophy in ways that embodied the virtues of both the Analytic and Chinese Traditions—e.g., combining the technical precision of Gödel's meta-mathematical approach to logic as well as the artistic and aesthetic perspicuity of Wittgenstein and Chinese philosophy. As an *analytically trained philosopher,* Wang was able to correct misconceptions of logic held by many Anglo-American philosophers due to the "guilt by association" in their minds of logic with positivism and the Vienna Circle, a view about the nature of logic that was not held by Gödel, who briefly attended the Circle. As a *mathematical logician*, Wang was able to provide a meta-mathematical perspective on logic to Chinese logicians. Wang quotes a mimeographed manuscript of Yueh-Lin Chin [1981] written in Kunming in

1943: "One of the features characteristic of Chinese philosophy is the underdevelopment of what might be called logico-epistemological consciousness."[52] As an *active researcher* in *mathematical logic* and *computation*, Wang was ahead of his time in proposing new graphical representations of computability, pioneering collaborations between academia and the emerging computer industry, and proposing new distinctions (e.g., feasible as opposed to theoretical computability). As an *Asian/American* academic, the trajectory, and limitations, of Wang's professional career sheds light on the unexamined practices of academic philosophy in the Anglo-American tradition, and the complex sociological dynamics that entered into the formation of the professional and personal lives of the "Thinking Orientals" of a previous generation and still enters into the formation of Asian American philosophers.

[52] Wang [1986], 193.

4.5 Wang's Empty Boat.

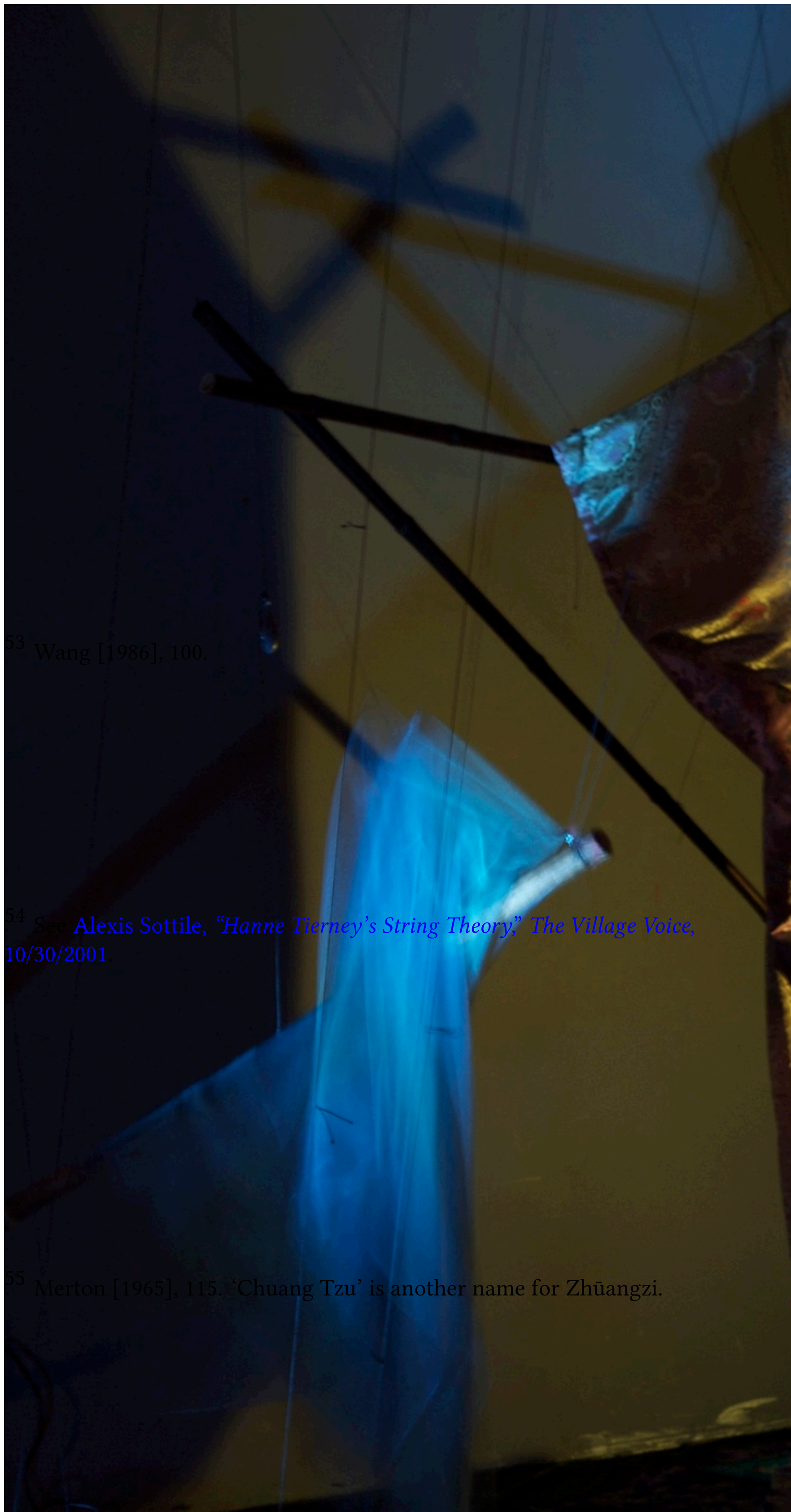
From 1991–1994, Juliet Floyd spent time in the company of Hao Wang and his wife Hanne Tierney and she notes Wang's return the conception of philosophy that had attracted him as a youth in China:

> Wang was inclined to consider artists with as much respect as he did scientists.... Wang's literary ambitions became stronger over time, and importantly shape his final writings. These bear an important relation to his ideas about "intuition" and his interest in Wittgenstein, whom he came to regard as 'art centered' rather than 'science centered' in his conception of philosophy.... Wang felt the literary effects of Wittgenstein's writing were not irrelevant to the content of his philosophy. I think we can assume that Wang felt the same way about his own books.... In the manner of Walter Benjamin, or perhaps better, of his Chinese forebears, Wang often proceeded by arranging quotations, without interpretation, in an effort to draw out the reader's reflection and response, thereby showing, but not himself directly stating.

One of Wittgenstein's most memorable sayings in the *Tractatus* is his famous ladder metaphor:

> 6.54 My propositions are elucidatory in this way: he who understands me finally recognizes them as senseless, when he has climbed out through them, on them, over them. (He must so to speak throw away the ladder, after he has climbed up on it.)

Wittgenstein's ladder, Wang noted, evokes the raft of the *Diamond Sūtra*: [53]



> The dharma I am preaching is analogous to a raft (which is to be discarded after use); even the dharma can be discarded, *a fortiori* the non-dharma.

Wang completed his last book *A Logical Journey: From Gödel to Philosophy* just months before his death s week before his 74th birthday, which was published posthumously in 1996.

In 2001 Hanne Tierney in collaboration with Wang's daughter and jazz musician Jane Wang created a performance piece "How Wang-Fo Was Saved"[54] at the *FiveMyles Gallery* in Crown Heights, Brooklyn.

The script for "How Wang-Fo Was Saved" was adapted from on a children's book by the Belgian-born French novelist and essayist Marguerite Yourcenar (1903–1987) that retells an ancient Chinese legend: Captured as he is walking with a disciple, the elderly painter Wang-Fo is brought before the Emperor. Spoiled by the beauty of Wang-Fo's paintings, the Emperor complains that nothing in reality is as beautiful as the artist's depictions. Wang-Fo is therefore to be executed for his lies. The Emperor commands Wang-Fo to finish painting one last canvas. Dutifully serving his sentence by completing his last canvas, Wang-Fo first paints a lake, then he draws a rowboat. As the waters rise and fill the Emperor's throne room, old Wang-Fo climbs into the boat and rows away. What is left behind is an empty boat.[55]

Who can free himself from achievement

And from fame, descend and be lost

[53] Wang [1986], 100.

[54] See Alexis Sottile, *"Hanne Tierney's String Theory," The Village Voice,* 10/30/2001.

[55] Merton [1965], 115. 'Chuang Tzu' is another name for Zhūangzi.

Amid the masses of men?

He will flow like Tao, unseen,

He will go about like Life itself

With no name and no home.

Simple is he, without distinction

To all appearances he is a fool.

His steps leave no trace. He has no power.

He achieves nothings, has no reputation.

Since he judges no one,

No one judges him

Such is the perfect man:

His boat is empty.

We shall end our reflections on Hao Wang's logical journey with a meditation on Wittgenstein's famous advice:

Wovon man nicht sprechen kann, darüber muss man schweigen.

Whereof one cannot speak, thereof one must pass over in silence.

Even Wittgenstein could not follow his own austere advice, so perhaps it is more fitting to conclude with the paradoxical humor and literary elegance of Zhūangzi:

> 荃者所以在魚，得魚而忘荃；蹄者所以在兔，得兔而忘蹄；言者所以在意，得意而忘言。
>
> 吾安得夫忘言之人而與之言哉！

The bamboo fish net exists for catching fish. Once the fish is caught, forget the net!

The rabbit snare exists for trapping rabbits. Once the rabbit is trapped, forget the snare!

Words exist because they are used for expressing meaning. Once the meaning is grasped, forget the words!

Where can I meet those who have forgotten words and have a word with them?[56]

Wang's many unsung contributions to the analytic tradition in philosophy and his life "chock full of contradictions" shed light not only on his own struggles but also on "*the difficult course of an entire generation and people.*" The goal of this article has been to explore how Wang's life could add to the conversation about the Analytic Tradition and Chinese Philosophy and to serve as a catalyst for diversifying what Wang once praised as "the elusive concept of an American spirit."[57]

4.6 TODO

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[56] I am indebted to Professor T. K. Chu for calling my attention to these poignant and paradoxical words of Zhūangzi.

[57] Wang [1986], 195, remarking on how despite the fact that Quine was influenced by mathematical logic and Dewey by Hegelian philosophy, there is nonetheless a "certain partial convergence of views." I wish to thank my colleague Ed Casey for giving me the opportunity to participate in the APA panel on Chinese Philosophy and the Analytic Tradition, organized by Professors Chung-ying Cheng and Linyu Gu. I also wish to thank Hanne Tierney for talking with me and for photographs. When I asked why she had agreed to meet sight unseen, Hanne with a sparkle in her eyes, replied: *"You had me at 'Hao'!"* In an email following up our conversation, Hanne wrote: *"It was such a pleasure to me to meet you and talk about Hao. He would have enjoyed it all himself very much, but he also would have so completely known what Gary was talking about...."*